1617-257 Wed Mar 22, hour 66: General Integration of Forms

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Read along: Sec 34-35 and 37. HW17 due, HW18 on web by midnight.

If M is oriented & α & β are positive, $\int_M^{\alpha} \omega = \int_M^{\beta} \omega =: \int_M \omega$ on board

practical def of $\int_M \omega$: By chopping in pieces w/ meas-0 exceptions.

Example Let S^2 be oriented as ∂D^3 and let $\omega \in \Omega^2(\mathbb{R}^3)$ be

$\omega = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$. compute $\int_{S^2} \omega = \int_{S^2} i^*\omega$.

$\alpha: [0,\infty)_r \times [0,2\pi]_\theta \times [-\frac{\pi}{2},\frac{\pi}{2}]_\phi$ be $\alpha(r,\theta,\phi) = (r\cos\theta\cos\phi, r\sin\theta\cos\phi, r\sin\phi)$

$\det D\alpha = r^2\cos\phi > 0$ so $\beta: [0,2\pi]_\theta \times [-\frac{\pi}{2},\frac{\pi}{2}]_\phi \to S^2$ by

$\beta(\theta,\phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$ is positive.

$\beta^* i^* \omega = \cos\theta\cos\phi(d\sin\theta\cos\phi)\wedge(d\sin\phi) + \sin\theta\cos\phi(d\sin\phi)\wedge d(\cos\theta\cos\phi) + \sin\phi(d\cos\theta\cos\phi)\wedge(d\sin\theta\cos\phi)$

$= d\theta\wedge d\phi(\cos^2\theta\cos^3\phi + \sin^2\theta\cos^3\phi + \sin^2\phi\sin^2\theta\cos\phi + \sin^2\phi\cos^2\theta\cos\phi)$

$= d\theta\wedge d\phi(\cos^3\phi + \cos\phi\sin^2\phi) = \cos\phi\, d\theta\wedge d\phi$

So $\int_M \omega = \int_{[0,2\pi]_\theta\times[-\frac{\pi}{2},\frac{\pi}{2}]_\phi} \cos\phi\, d\theta d\phi = 4\pi$

Assuming M is compact, or other cond. are specified.

Theoretical def of $\int_M \omega$: Find a PoU ϕ_i subordinate to "positive charts of M"

* $\phi_i \in C^\infty$ * $\operatorname{supp}\phi_i \subset$ pos. chart U_i * loc. finite * $\sum\phi_i = 1$

and set $\int_M^{\phi} \omega := \sum_{i\in I} \int_M \phi_i\omega$

Is this the same as $\int_M^{\psi} \omega = \sum_{j\in J} \int_M \psi_j\omega$?

$\int_M^{\phi} \omega = \sum_i \int_M \phi_i\omega = \sum_i \int_M \sum_j \psi_j\phi_i\omega = \sum_i\sum_j \int_M \psi_j\phi_i\omega =$

$= \sum_j\sum_i \int_M \phi_i\psi_j\omega = \sum_j \int_M \sum_i \phi_i\psi_j\omega = \sum_j \int_M \psi_j\omega = \int_M^{\psi}\omega$

It is easy to show that 1. $\int(a\omega_1 + b\omega_2) = a\int\omega_1 + b\int\omega_2$.

2. $\int_{-M} \omega = \int_M -\omega$

done line

Thm If M^k is compact and oriented and $\omega \in \Omega^{k-1}(M)$, then $\int d\omega = \int \omega$

$M \quad \partial M$

Proof 1. Interior charts $\rightsquigarrow \lambda \in \Omega^{k-1}(\mathbb{R}^k)$ w/ $\operatorname{supp}\lambda \subset \operatorname{int} Q$

$$\lambda = \sum \lambda_i \, dx_1 \ldots \widehat{dx_i} \, dx_k \qquad d\lambda = \sum (-1)^{i-1} \frac{\partial \lambda_i}{\partial x_i} dx_k$$

$$\int_Q d\lambda = \sum_{i=1}^{k} (-1)^{i-1} \int_Q \frac{\partial \lambda_i}{\partial x_i} = 0$$

2. Bndry charts: $\lambda \in \Omega^{k-1}(\overset{\nearrow H_1^k}{\mathbb{R}^k_{x_1 \geq 0}})$ $\operatorname{supp}\lambda \subset \operatorname{int}_{H_1^k} Q = [0, b_1] \times \overbrace{\prod_{i=2}^{k} [a_i, b_i]}^{\in Q'}$

$$\int_Q d\lambda = \sum_{i=1}^{k} (-1)^{i-1} \int_Q \frac{\partial \lambda_i}{\partial x_i} = \int_Q \frac{\partial \lambda_1}{\partial x_1} = -\int_{x' \in Q'} \lambda_1(0, x') = -\int_{\{0\} \times Q'} \lambda_1 \, dx_2 \wedge \ldots dx_k = -\int_{\{0\} \times Q'} \lambda = \int_{\partial Q} \lambda$$

3. $\omega = \sum \phi_i \omega$

$$\int_{\partial M} \omega = \sum_i \int_{\partial M} \phi_i \omega = \sum_i \int_M d(\phi_i \omega) = \sum_i \int_M d\phi_i \wedge \omega + \phi_i \, d\omega = \int_M \left(d \sum \phi_i\right) \wedge \omega + \left(\sum \phi_i\right) \wedge d\omega$$

$$= \int_M d\omega$$