1617-257 Wed Mar 15, hour 63: preliminary integration, orientations

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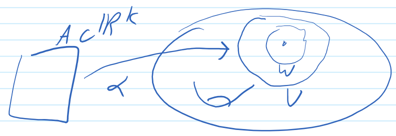
Read along: Sec 33-35. HW 17 on web by midnight!

$M^k \subset \mathbb{R}^n$, $\omega \in \Omega^{top}(M)$, $\operatorname{supp}\omega \subset U = \operatorname{im}\alpha$, $\alpha: A \subset \mathbb{R}^k \to M$ a patch

Define $\displaystyle \overset{\alpha}{\underset{M}{\not\int}} \omega = \int_{\mathbb{R}^k} \alpha^*\omega = \int_{\mathbb{R}^k} f$, where $\alpha^*\omega = f\,dx_{\underline{k}}$

preliminary



Claim If $\beta: \mathbb{R}^k \to V \subset \mathbb{R}^n$ is a patch also containing $\operatorname{supp}\omega$, (and $U \cap V$ is connected)

then $\displaystyle \overset{\alpha}{\underset{M}{\not\int}} \omega = \pm \overset{\beta}{\underset{M}{\not\int}} \omega$ our next battle, "orientations".

on board

Pf Assume also $\operatorname{supp}\omega \subset V = \operatorname{im}\beta$, $\beta: B \subset \mathbb{R}^k \to M$ a patch, and let $\phi = \beta^{-1}\circ\alpha$; it is C^r. Then $\displaystyle \overset{\beta}{\underset{M}{\not\int}} \omega = \int_{\mathbb{R}^k} g$ where $\beta^*\omega = g\,dx_{\underline{k}}$

$$\overset{\alpha}{\underset{M}{\not\int}} \omega = \int_A \alpha^*\omega = \int_{A' = \alpha^{-1}(U\cap V)} (\beta\circ\phi)^*\omega = \int_{A'} \phi^*\beta^*\omega = \int_{A'} \phi^*(g\,dx_{\underline{k}}) = \int_{A'} (g\circ\phi)\cdot\det D\phi \cdot dx_{\underline{k}}$$

$$= \pm \int_{A'} g\circ\phi\,|\det D\phi| = \pm \int_{B' = \beta^{-1}(U\cap V)} g = \pm \overset{\beta}{\underset{M}{\not\int}} \omega \qquad \square$$

A story about right hands in Canada, India, Australia.

Definition An orientation $\mathcal{O}$ on a f.d. v.s. V is a choice of an ordered basis for V, regarded up to positive-det changes of bases:

$$(v_1 \dots v_n) \underset{\text{"is same as"}}{\sim} (u_1 \dots u_n) \quad \text{if } \det(C_v^u) > 0.$$

should have been: orientation is a "sign" map $\nu: \{\text{ordered bases}\} \to \pm 1$ s.t. $\nu(v) = \nu(u)\operatorname{sign}\det C_v^u$

An oriented v.s. is a v.s. along w/ a choice of an orientation on it.

Claim Every f.d. v.s. V has exactly two distinct orientations.

Minor properties: 1. reversing any basis vector reverses the orientation.
2. Swapping any two basis vectors -//-.

done live

Def An orientation on a manifold M is a cont. choice of orientations $\mathcal{O}_x$ on T_xM, for every $x \in M$.

Cont: Every $p \in M$ has a nbd W with cont. vector fields $u_1 \dots u_k$ on W s.t. for every $x \in U$, $(u_1(x), \dots u_k(x)) \sim \mathcal{O}_x$.

Examples:

1. If $M^k \subset \mathbb{R}^{k+1}$ and $\mathbb{R}^{k+1}$ is oriented, then there is a bijection

$$\left\{\begin{matrix}\text{orientation of} \\ M\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Cont. varying choice of unit} \\ \text{normal vectors to } M\end{matrix}\right\}$$

$\Big(:= \forall p \in M$ a vector $n(p) \in T_p\mathbb{R}^{k+1}$ s.t.
1. $n(p) \perp T_pM$
2. $p \mapsto n(p)$ is cont. $\Big)$

Pf...

2. $p \mapsto n(p)$ is cont.

2. If M is oriented, so is ∂M.