1617-257 Wed Jan 25, hour 45: Manifolds and their boundaries

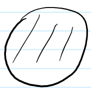
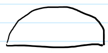
$$\int_M d\omega = \int_{\partial M} \omega$$

January 16, 2017 8:21 AM

- TT2 returns at end.
- HW11 due, HW12 on web by midnight.
- Read Along: Still 23-24.
- Riddle Along: n red points and n blue points are placed in the plane with no 3 on the same line. Prove that it is possible to pair them up using n straight line segments, so that no two of these segments will intersect.

k-manifold of class C^r in $\mathbb{R}^n$: $M\subset\mathbb{R}^n$, for all $p\in M$ $\exists$ nbd V of p, $q\in H^k$, nbd U of q, $\alpha: U\longrightarrow V$ homeomorphism, C^r, $D\alpha$ of max rank, $\alpha(p)=q$.
If $q\in\partial H^k=\mathbb{R}^{k-1}\times\{0\}$, p is in ∂M.

on board

Q Is  homeo to 

Precisely, is $B^k=\{x\in\mathbb{R}^k: \|x\|<1\}$ homeo to $HB^k=\{x\in B^k: x_k\geq 0\}$? (No)

2 For $k=1$ use connectedness.

4 For $k=2$ use simple-connectedness

6 For $k>2$ very hard.

Aside Is  homeo to ?

Is B^k homeo to B^n, for $n\neq k$? (No)

1 For $k=1$, $n>1$ use connectedness.

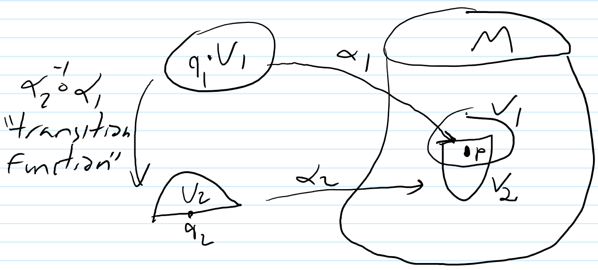
3 For $k=2$, $n>2$ use simple-connectedness.

5 For $k>2$, $n>k$ very hard.

What if we replace homeo w/ diffeo [homeo + diffable + inverse is diffable]?

8 Nearly trivial.

7 Trivial.

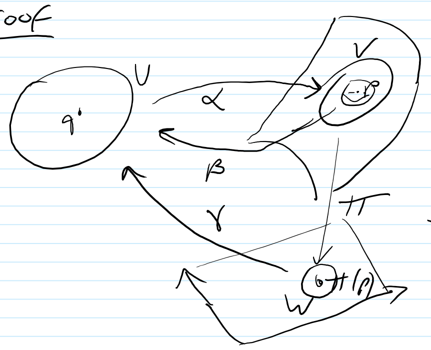


$V_1\cap V_2$ in an open nbd of p in M
Let $U_1'=\alpha_1^{-1}(V_1\cap V_2)$ $U_2'=\alpha_2^{-1}(V_1\cap V_2)$
these are open nbds of q_1, q_2 in H^k
$\alpha_2^{-1}\circ\alpha_1: U_1'\longrightarrow U_2'$ is a homeo mapping an inner point to an edge point; this is impossible but very hard to show. If only we knew α_1^{-1} & α_2^{-1} were diffable.... We'll only do α_1^{-1}; α_2^{-1} is similar (and is in book)

Proposition If M^k is a class C^r manifold in $\mathbb{R}^n$ and $\alpha: U\subset\mathbb{R}^k\longrightarrow V\subset M^k$ is a coordinate patch, then $\alpha^{-1}: V\longrightarrow U$ is C^r.

done line

Proof



NTS: there is a C^r function $\beta: V'\to U'$, s.t. V' is a nbd of p in $\mathbb{R}^n$, U' is a nbd of q contained in U, and $\beta\circ\alpha=Id$ on U'.

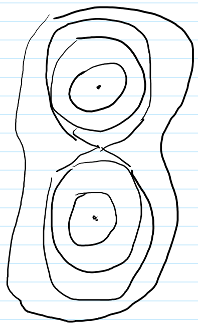
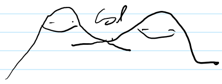
Pf: For convenience, assume the first k rows of $d\alpha(q)$ are lin. indep., and let $\pi:\mathbb{R}^n\to\mathbb{R}^k$ be the proj. on the first k coords. Then $d(\pi\circ\alpha)(q)$

is invertible, so $\pi\circ\alpha$ has an inverse γ near $\pi(p)$; precisely, on some nbd W of $\pi(p)$ for which $(\pi\circ\alpha)^{-1}(W)\subset U$.

Now let $V'=\pi^{-1}(W)$, $U'=(\pi\circ\alpha)^{-1}(W)=\alpha^{-1}(M\cap V')$ and $\beta=\gamma\circ\pi$.

Corollary "transition functions" are C^r.

Theorems we skip:

1. ∂M is a $(k-1)$-manifold.
2.   If $A\subset\mathbb{R}^n$ is open and $f:A\to\mathbb{R}$ is C^r, ~~then for most h $f^{-1}(h)$ is a manifold.~~ and $h\in\mathbb{R}$ is such that whenever $p\in f^{-1}(h)$, $Df(p)$ has rank 1, then $f^{-1}(h)=N$ is a manifold, and so are $f^{-1}((-\infty,h])=M_1$ & $f^{-1}([h,\infty))=M_2$, and $\partial M_1=\partial M_2=N$.

Corollary: S^{n-1} is a mfld & $S^{n-1}=\partial D^n$.