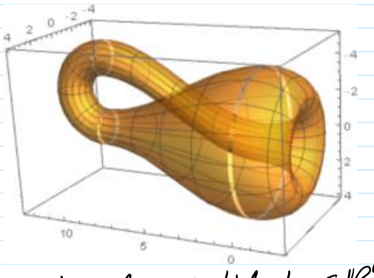
1617-257 Wed Jan 18, hour 42: Manifolds in Rn

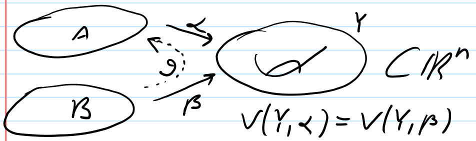
January 16, 2017 8:21 AM

- Riddle Along: On ZxZ, a visible roach R starts at (0,0) and once a minute jumps to the northeast, up to a distance of 10. Meanwhile, an exterminator E can poison one grid point per minute, away from R. Can E trap R?
- Nothing to say about the Term Test.
- HW11 will be on web by midnight!
- Read Along: Sections 23 & 24.

$V(Y)=V(Y,\alpha):=\int_A V(D\alpha):=\int_A |\det (D\alpha)^T D\alpha|^{1/2}$



the Klein bottle $K\subset\mathbb{R}^4$



$V(Y,\alpha)=V(Y,\beta)$

precisely, if $g:B\to A$ is a diffeomorphism of open sets in $\mathbb{R}^k$, and $\alpha:A\to\mathbb{R}^n$ is a manifold, set $\beta=\alpha\circ g$ and then $\alpha(A)=\beta(B)=Y$ and $V(Y,\alpha)=V(Y,\beta)$.

start line

Mild generalization: If f is a cont. function on Y, define

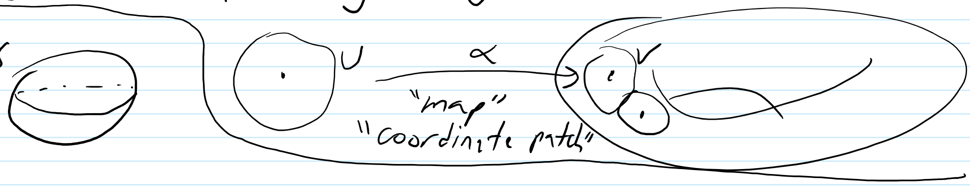
$$\int_Y f\,dV := \int_A (f\circ\alpha)V(D\alpha)$$

This too is invariant under re-parametrization.

$\int_M d\omega=\int_{\partial M}\omega$ M: "A nice k smooth subset of $\mathbb{R}^n$" = sphere, torus, $\overset{\mathbb{R}^4}{\underset{}{K}}$, $O(n)\subset\mathbb{R}^{n^2}$

Def a k-manifold (w/o bndry, of class C^r) in $\mathbb{R}^n$ is $M\subset\mathbb{R}^n$ s.t. each $p\in M$ has an open nbd V s.t. there is an open $U\subset\mathbb{R}^k$ & a C^r homeomorphism $\alpha:U\to V$ whose differential has rank k for every $x\in U$

Example: Discuss

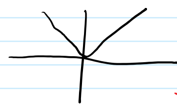


(non-) examples:

1. $\alpha:\mathbb{R}\to\mathbb{R}^2$ by $\alpha(t)=(t^3,t^2)$:

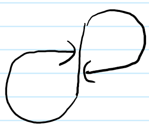
$(d\alpha)(0)=0$ ✗

2. $\beta(t)=(t^3,|t|^3)$ is a homeomorphism, yet no good.



I should have merged these two.

3. $\alpha:\mathbb{R}\to\mathbb{R}^2$ by



done line

done line

Wish to define "mflds w/ bndry" & "bndry ∂M of a mfld M"

Warning: not the same as $Bd\,A$, w/ $A\subset\mathbb{R}^n$!

<u>Premature</u> <u>Definition</u> Let $H^k=\{x\in\mathbb{R}^k : x_k\geq 0\}$

A k-manifold (of class C^r, possibly w/ bndry) in $\mathbb{R}^n$ is $M\subset\mathbb{R}^n$ s.t. each $p\in M$ has an open nbd V s.t. there is an open $U\subset H^k$ & a C^r homeomorphism $\alpha: U\to V$ whose differential has rank k for every $x\in U$.

The bndry ∂M of M is $\partial M=\left\{p\in M : \begin{array}{l}\text{for some patch } \alpha,\\ p=\alpha(q) \text{ w/ } q\in\partial H^k=\mathbb{R}^{k-1}\times\{0\}\end{array}\right\}$

It is a manifold (of class C^r, w/ no bndry) of dim $(k-1)$.

Issues: (I1) What does diffability mean of H^k? What's $d\alpha$ on the edge of H^k?

(I2) Is ∂M well-defined? More precisely, is $\partial M=M$?

(I3) Are we sure ∂M is a manifold?

Dispatch (I1):

<u>Def</u> Let $S\subset\mathbb{R}^k$ & $f:S\to\mathbb{R}^n$. We say that f is a class C^r on S if there is an open $U\supset S$ & a C^r $g:U\to\mathbb{R}^n$ s.t. $g|_S=f$.

<u>Easy Fact</u>: If $f:H^k\to\mathbb{R}^n$ is C^r, then $(Df)(p)$ makes sense for $p\in\partial H^k=\mathbb{R}^{k-1}$; namely, all extentions of f to $U\supset H^k$ have the same differential at p.

<u>Surprisingly hard Fact</u>: "Diffability on S" is a local property. Namely, if $f:S\to\mathbb{R}^n$ is such that every $p\in S$ has a nbd U_p in $\mathbb{R}^k$ s.t. $f|_{S\cap U_p}$ is C^r, then f is C^r.