1617-257 Wed Jan 11, hour 39: Gram-Schmidt, k-vols in Rn (linear case)

January 8, 2017 3:25 PM

TT: Tue Jan 17 5PM-7PM @ EX 300. Extra OH: Dror Mon Jan 16 5:30-8 BA 6178, Jeff Tue Jan 17 11-2 Huron 215 10th floor.
Approximate Details:

- Material: Everything from last TT / HW6 until Friday, roughly proportional to time spent + around 20% from older material.
- Roughly choose 4/5, some questions multi-part.
- About 1/3 "prove as in class", 1/3 "solve as in HW", 1/3 "solve fresh".
- How I used to prepare.

Read along: Sec 21.
Riddle Along: Cars A,B,C,D drive in the Sahara desert on generic straight lines and at constant speed; it is known that A meets B (they arrive at the same place at the same time), A meets C, A meets D, B meets C, and B meets D. Does C necessarily meet D?

Def: $h:\mathbb{R}^n\to\mathbb{R}^n$ is an "isometry" if $\forall x,y$ $d(h(x),h(y))=d(x,y)$ (Euc)

Thm h is an isometry iff it is of the form
$h(x)=p+Ax$, where $A\in M_{n\times n}$ satisfies $A^TA=I$

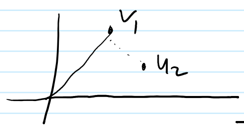
Already know: WLOG, $h(0)=0$; h preserves norms & dot products,
$A:=(h(e_1)\,|\,h(e_2)\,|\,\dots\,|\,h(e_n))\in O(n)$ $[A^TA=A]$ on board

Claim $h(\sum x_ie_i)=\sum x_i h(e_i)$ so $h(x)=Ax$

Pf Let $\Delta=h(\sum x_ie_i)-\sum x_ih(e_i)$. Then $\langle\Delta,h(e_j)\rangle=0$,
so $\Delta Ae_j=0$ so $\Delta A=0$ so $\Delta=0$. $\square$

Important Aside ("The Gram-Schmidt" process") If $\{u_i\}$ is a basis of an inner-product space V [for this class, okay to restrict to $V=\mathbb{R}^n$, w/ usual inner product], then there is an (almost unique) orthonormal basis $\{v_i\}$ s.t. $\exists k$ $\mathrm{span}\{v_i:1\le i\le k\}=\mathrm{span}\{u_i:1\le i\le k\}$

Example $u_1=\begin{pmatrix}3\\4\end{pmatrix}$ $u_2=\begin{pmatrix}1\\1\end{pmatrix}$ $v_1=\begin{pmatrix}3/5\\4/5\end{pmatrix}$



$v_2'=u_2-\langle v_1,u_2\rangle v_1$
$=\begin{pmatrix}1\\1\end{pmatrix}-\frac{7}{5}\begin{pmatrix}3/5\\4/5\end{pmatrix}$
$=\begin{pmatrix}4/25\\-3/25\end{pmatrix}$ $v_2=\begin{pmatrix}4/5\\-3/5\end{pmatrix}$

In general

$v_1'=u_1$ $\qquad v_1=v_1'/\|v_1'\|$

$v_2'=u_2-\langle v_1,u_2\rangle v_1$ $\qquad v_2=v_2'/\|v_2'\|$

$v_3'=u_3-\langle v_1,u_3\rangle v_1-\langle v_2,u_3\rangle v_2$ $\qquad v_3=v_3'/\|v_3'\|$

$v_k'=u_k-\sum_{i=1}^{k-1}\langle v_i,u_k\rangle v_i$ $\qquad v_k=v_k'/\|v_k'\|$

proof that this works: Exercise.

Thm There's a unique $V:(\mathbb{R}^n)^k\to\mathbb{R}_{\ge0}$ s.t.

1. If $h:\mathbb{R}^n\to\mathbb{R}^n$ is an orthogonal trans, & $x_i\in\mathbb{R}^n$, then
$$V(h(x_1),\dots,h(x_k)) = V(x_1\dots x_k)$$
2. If $x_i\in\mathbb{R}^k\times\{0\}$, so $x_i=\begin{pmatrix}y_i\\0\end{pmatrix}$ w/ $y_i\in\mathbb{R}^k$, then
$$V(x_1\dots x_k) = |\det(y_1|\dots|y_k)|$$

done line

Furthermore, V vanishes iff $\{x_i\}$ are dependent, and
$$V(x_1\dots x_k) = |\det \underbrace{X^T\cdot X}_{k\times k}|^{1/2} \quad \text{w/ } X=(x_1|\dots|x_k)\in M_{n\times k}$$

<u>Pf</u> 1. Uniqueness: Given $x_1\dots x_k$, I'd like to tell you how to compute $V(x_1\dots x_k)$ using just 1–2. Find an O.N basis $\{f_i\}_{i=1}^{l}$ of the subspace W generated by $x_1\dots x_k$ (so $l\le k$) and extend it to an O.N. basis $\{f_i\}_{i=1}^{n}$ of $\mathbb{R}^n$. Let $g:\mathbb{R}^n\to\mathbb{R}$ be defined by $g(e_i)=f_i$. It is O.N., and so is $h:=g^{-1}$; $h(f_i)=e_i$. Furthermore, $h(W)=\text{span}\{h(f_i)\}_{i=1}^{l}=\text{span}\{e_i\}_{i=1}^{l}\subset\mathbb{R}^k\times\{0\}$, so $h(x_i)\in\mathbb{R}^k\times 0$, so 2&1 determine V.

2. Set $V(x_1\dots x_k)=|\det X^TX|^{1/2}$ & prove properties:

1. $V(h(x_1),\dots h(x_k)) = V(Ax_1\dots Ax_k) = |\det (AX)^TAX)|^{1/2} = |\det X^TX|^{1/2}$
2. If $x_1\dots x_k\in\mathbb{R}^k\times\{0\}$, $X=\begin{pmatrix}Y\\0\end{pmatrix}$ so
$$|\det(X^TX)|^{1/2} = |\det(Y^TY)|^{1/2} = |\det(Y)| \text{ as } Y \text{ is square.}$$

3. $\{x_i\}$ dep. $\Leftrightarrow$ V vanishes:

$\Rightarrow$ x_i dep $\Rightarrow \exists a\neq 0\ Xa=0 \Rightarrow X^TXa=0 \Rightarrow \det(X^TX)=0$.

$\Leftarrow$ $\det(X^TX)=0 \Rightarrow \exists a\neq 0\ X^TXa=0 \Rightarrow a^TX^TXa=0 \Rightarrow Xa=0$.

The 2C3 case: Improvize.