1617-257 Wed Feb 1, hour 48: multilinear algebra

January 16, 2017 8:21 AM

HW12 due, HW13 on web.
Read Along. Sections 26-27.
Riddle Along. Four points form a perfect square in the plane. Prove that you cannot turn that square into a bigger one by a sequence of moves of the form (x,y)->(x,2x-y) applied to pairs of points within the four.

Def. $\mathcal{L}^k(V) := \left\{ f : V^k \to \mathbb{R} : \begin{array}{l} v \mapsto f(v_1 \dots v_{i-1}, v, v_{i+1} \dots v_k) \text{ is} \\ \text{linear in } v, \text{ for all } 1 \le i \le k \end{array} \right\}$

on board

For $k=1$, this is V^*, "the dual of V". In general, these are "k-tensors".

Claim $\mathcal{L}^k(V)$ is a vector space.

Claim If $(a_1 \dots a_n)$ is a basis for V, and $I = (i_1 \dots i_k) \in \{1, \dots, n\}^k = \underline{n}^k$ then there is a unique $\phi_I \in \mathcal{L}^k(V)$ s.t. for every $J = (j_1 \dots j_k) \in \underline{n}^k$

$$(*) \qquad \phi_I(a_{j_1} \dots a_{j_k}) = \begin{cases} 1 & I = J \\ 0 & \text{otherwise} \end{cases} = \delta_{IJ}$$

These ϕ_I make a basis of $\mathcal{L}^k(V)$ (so $\dim \mathcal{L}^k(V) = n^k$)

steps. 1. An element of $\mathcal{L}^k(V)$ is determined by it's values on seqs. of basis vectors, so if exists, ϕ_I is unique.

2. If $k=1$, $\phi_i(a_j) = \delta_{ij}$ determines $\phi_i \in \mathcal{L}^1(V) = V^*$.

3. Set $\phi_I(v_1 \dots v_k) = \phi_{i_1}(v_1) \cdot \phi_{i_2}(v_2) \dots \phi_{i_k}(v_k)$; it is in $\mathcal{L}^k(V)$ and it satisfies $(*)$.

4. The ϕ_I's are lin-indep.

done line

5. The ϕ_I's span $\mathcal{L}^k(V)$.

Easy yet worth noting: There is a map $\otimes : \mathcal{L}^k(V) \times \mathcal{L}^m(V) \to \mathcal{L}^{k+m}(V)$. It is bilinear, associative, & $\phi_I = \phi_{i_1} \otimes \phi_{i_2} \otimes \dots \phi_{i_k}$

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points, tangent vectors push forward; functions pull back

Important yet often under-rated: Given a linear $T: V \to W$, there is a "pullback" $T^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$. It is

* linear (meaning...)
* compatible w/ $\otimes$ (meaning, ...)
* contravariant: (meaning: ....)