# 1617-257 Wed Apr 5, hour 72: Maxwell

April 3, 2017 9:41 AM

Evals: 49/92. HW19 due! Final exam notes on web!
The follow http://drorbn.net/AcademicPensieve/Classes/1617-257a-AnalysisII/160912/Day1.pdf

Prelims: 1. $\omega\in\Omega^k(\mathbb{R}^n)$, $d\omega=0 \Rightarrow \exists\lambda\in\Omega^{k-1}$ s.t. $d\lambda=\omega$

2. $\int \omega\wedge d\eta = -(-1)^{\deg\omega}\int (d\omega)\wedge\eta$ if supp $\omega/\eta$ is compact.

3. $*dx_I = \pm dx_{I^c}$ s.t. $\omega\wedge *\eta = \langle\omega,\eta\rangle dx_n$

4. A least action principle for $F=ma$: omitted. on board

**The Action Principle.** The 4 *Vector* ~~*Field*~~ Potential is a compactly supported 1-form $A$ on $\mathbb{R}^4$ which extremizes the *action*

$$\boxed{S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2}\|dA\|^2\, dtdxdydz + J\wedge A}$$

where the 3-form $J$ is the *charge-current*.

$f$ has crit. pt. at $b \Rightarrow \frac{\partial}{\partial\epsilon} f(a+\epsilon b)\big|_{\epsilon=0}=0$ $(\Rightarrow df_a=0)$

so take $A,B\in\Omega^1_c(\mathbb{R}^4)$ &

$$\frac{\partial}{\partial\epsilon}S_J(A+\epsilon B)\Big|_{\epsilon=0} = \frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\int_{\mathbb{R}^4}\frac{1}{2}\langle dA+\epsilon dB, dA+\epsilon dB\rangle + J\wedge(A+\epsilon B)$$

$$= \frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\int_{\mathbb{R}^4}\frac{1}{2}\langle dA,\epsilon dB\rangle + \frac{1}{2}\langle\epsilon dB, dA\rangle + \epsilon J\wedge B = \int_{\mathbb{R}^4}\langle dB,dA\rangle + J\wedge B$$

$$= \int_{\mathbb{R}^4} dB\wedge *dA + J\wedge B = \int_{\mathbb{R}^4} *dA\wedge dB + J\wedge B = \int_{\mathbb{R}^4}(-d*dA+J)\wedge B \Longrightarrow d*dA = J$$

**The Euler-Lagrange Equations** in this case are $d\star dA = J$, meaning that there's no hope for a solution unless $dJ=0$, and that we might as well (think Poincaré's Lemma!) change variables to $F := dA$. We thus get

$$\boxed{dJ=0 \qquad dF=0 \qquad d\star F = J}$$

**These are the Maxwell equations!** Indeed, writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

$$F = E_x dxdt + E_y dydt + E_z dzdt + B_x dydz + B_y dzdx + B_z dxdy$$

$E$: "Electric Field" $\quad B$: "magnetic Field"

$$*F = -B_x dxdt - B_y dydt - B_z dzdt - E_x dydz - E_y dzdx - E_z dxdy$$

| | | | |
|---|---|---|---|
| $dJ=0 \Longrightarrow$ | $\frac{\partial\rho}{\partial t} + \operatorname{div} j = 0$ | "conservation of charge" | $dt\,dx\,dy\,dz$ |
| $dF=0 \Longrightarrow$ | $\operatorname{div} B = 0$ | "no magnetic monopoles" | $dx\,dy\,dz$ |
| | $\operatorname{curl} E = -\frac{\partial B}{\partial t}$ | that's how generators work! | $dt\,dy\,dz$ etc. |
| $d*F=J \Longrightarrow$ | $\operatorname{div} E = -\rho$ | "electrostatics" | $dx\,dy\,dz$ |
| | $\operatorname{curl} B = -\frac{\partial E}{\partial t} + j$ | that's how electromagnets work! | $dt\,dy\,dz$ etc. |

**Exercise.** Use the Lorentz metric to fix the sign errors.

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**Exercise.** With $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ use $S = mc\int_{e_1}^{e_2}(ds + eA)$ to derive Feynman's "law of motion" and "force law".