1617-257 Mon Jan 9, Hour 38: Isometries, Gram-Schmidt, k-vols in Rn

December 5, 2016 8:56 AM

TT2: Much like TT1, details on Wed.
Agenda: Isometries, Gram Schmidt, k-volumes in R^n.
Read along: Sec 20, 21.
Riddle along: A&B play. A writes the numbers 1 through 18 on three blank dice; B chooses on of the 3, A chooses one of the remaining 2, and they discard the 3rd. They then play 1000 games of "dice war", on real money. Whom would you rather be, A or B?

Def: $h:\mathbb{R}^n\to\mathbb{R}^n$ is an "isometry" if $\forall x,y$ $d(h(x),h(y))=d(x,y)$ (Euc)

Thm h is an isometry iff it is of the form
$h(x)=p+Ax$, where $A\in M_{n\times n}$ satisfies $A^TA=I$

Comments 1. Such h is "volume preserving".

on board

2. Such h is a "rotation followed by translation".
[A l.t. is a "rotation" if $A^TA=I$; alt., if it is "orthogonal", meaning it's columns form an "orthonormal basis"]
3. Rotation matrices $O(n):=\{A: A^TA=I\}$ form a "group":
   0. $A,B\in O(n)\Rightarrow A\cdot B\in O(n)$.
   1. $(AB)C=A(BC)$
   2. $\exists I\in O(n)$ s.t. $AI=IA=A$.
   3. $\forall A\in O(n)$ $\exists B\in O(n)$ $AB=BA=I$ [use, e.g. $\det(A)=\pm1$]

proof of thm $\Leftarrow$: easy (though write in full....; $\|x\|^2=x^T\cdot x\ldots$)

$\Rightarrow$ steps: 1. WLOG, $h(0)=0$.
2. h preserves norms.
3. using $\|x\cancel{+}y\|^2=\ldots$, h preserves dot products.
4. set $A=(h(e_1)|\ldots|h(e_n))$; then $A\in O(n)$.

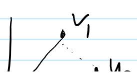
−, not +!
so $\|h(x)-h(x)\|^2=d(h(x),h(y))^2$

done line

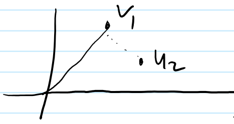
5. Claim $h(\sum x_ie_i)=\sum x_i h(e_i)$ so $h(x)=Ax$

Pf Let $\Delta=h(\sum x_ie_i)-\sum x_i h(e_i)$. Then $\langle\Delta,h(e_j)\rangle=0$,
So $\Delta Ae_j=0$ so $\Delta A=0$ so $\Delta=0$.

Important Aside ("The Gram-Schmidt" process") If $\{u_i\}$ is a basis of an inner-product space V [For this class, okay to restrict to $V=\mathbb{R}^n$, w/ usual inner product], then there is an (almost unique) orthonormal basis $\{v_i\}$ s.t. $\exists k$ $\operatorname{span}\{v_i: 1\le i\le k\}=\operatorname{span}\{u_i: 1\le i\le k\}$

Example $u_1=\begin{pmatrix}3\\ \ldots\end{pmatrix}$ $u_2=\begin{pmatrix}1\\ \ldots\end{pmatrix}$ $v_1=\begin{pmatrix}3/5\\ \ldots\end{pmatrix}$  $v_2'=u_2-\langle v_1,u_2\rangle v_1$

Example $u_1 = \begin{pmatrix}3\\4\end{pmatrix}$ $u_2 = \begin{pmatrix}1\\1\end{pmatrix}$ $v_1 = \begin{pmatrix}3/5\\4/5\end{pmatrix}$



$$v_2' = u_2 - \langle v_1, u_2\rangle v_1 = \begin{pmatrix}1\\1\end{pmatrix} - \frac{7}{5}\begin{pmatrix}3/5\\4/5\end{pmatrix} = \begin{pmatrix}4/25\\-3/25\end{pmatrix} \qquad v_2 = \begin{pmatrix}4/5\\-3/5\end{pmatrix}$$

In general

$$v_1' = u_1 \qquad v_1 = v_1'/\|v_1'\|$$

$$v_2' = u_2 - \langle v_1, u_2\rangle v_1 \qquad v_2 = v_2'/\|v_2'\|$$

$$v_3' = u_3 - \langle v_1, u_3\rangle v_1 - \langle v_2, u_3\rangle v_2 \qquad v_3 = v_3'/\|v_3'\|$$

$$v_k' = u_k - \sum_{i=1}^{k-1}\langle v_i, u_k\rangle v_i \qquad v_k = v_k'/\|v_k'\|$$

proof that this works: Exercise.

Thm There's a unique $V : (\mathbb{R}^n)^k \longrightarrow \mathbb{R}_{\geq 0}$ s.t.

1. If $h : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal trans, & $x_i \in \mathbb{R}^n$, then
$$V(h(x_1), \dots, h(x_k)) = V(x_1 \dots x_k)$$
2. If $x_i \in \mathbb{R}^k \times \{0\}$, so $x_i = \begin{pmatrix}y_i\\0\end{pmatrix}$ w/ $y_i \in \mathbb{R}^k$, then
$$V(x_1 \dots x_k) = |\det(y_1 | \dots | y_k)|$$

Furthermore, V vanishes iff $\{x_i\}$ are dependent, and

$$V(x_1 \dots x_k) = |\det X^T \cdot X|^{1/2} \qquad \text{w/ } X = (x_1 | \dots | x_k) \in M_{n\times k}$$

Pf Improvize.