1617-257 Mon Jan 30, hour 47: Integration on manifolds, multilinear algebra

January 16, 2017 8:21 AM

Meta-riddle:

2	7	6
9	5	1
4	3	8

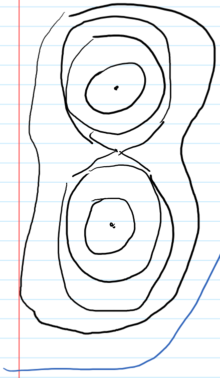
which two riddles I have posed does a magic square solve?

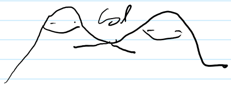
You are less lost than you realize!

Agenda. Two sketches; then multi-linear algebra.

TT2: Today is appeals deadline, right after class.

Read Along: Sec 25, 26





If $A \subset \mathbb{R}^n$ is open and $f: A \to \mathbb{R}$ is C^r, then for most h $f^{-1}(h)$ is a manifold.

on board.

Precisely,
If $h \in \mathbb{R}$ is such that whenever $p \in f^{-1}(h)$, $df(p)$ has rank 1, then $f^{-1}(h) = N$ is a manifold, and so are $f^{-1}((-\infty, h]) = M_1$ & $f^{-1}([h, \infty)) = M_2$, and $\partial M_1 = \partial M_2 = N$

Corollary: S^{n-1} is a mfld & $S^{n-1} = \partial D^n$.

Sketch of proof: Implicit Function theorem classic!

(Narrow-minded) Integration of Functions on Compact Manifolds: $\int_M f\,dV = \int_V (f \circ \alpha) V(D\alpha)$

* Makes sense immediately if function is supported on one patch [that is, if supported on one patch in two ways, both ways give same answer.]
* Can be defined using Po$\mathbb{1}$.
* Indep of choices.

In practice chop up M to pieces w/ meas-0 intersection, integrate on each piece.

The Partitions of Unity Lemma. Given a collection $\mathcal{A}$ of open sets in $\mathbb{R}^k$ whose overall union is $A = \bigcup_{U \in \mathcal{A}} U$, there exists a sequence $\{\phi_i\}$ of non-negative compactly-supported C^∞ functions such that:

1. For each i there is some $U \in \mathcal{A}$ such that $\operatorname{supp}(\phi_i) \subset U$.
2. Every $x \in A$ has a neighborhood V such that $\{i : V \cap \operatorname{supp}(\phi_i) \neq \emptyset\}$ is finite.
3. $\sum_{i=1}^{\infty} \phi_i = 1$ on A.

Multi-linear Algebra.

* Motivation: "$V(d\alpha)$" is narrow minded
  We need "something" that on k "tangent vectors" would be:
  1. multi-linear.
  2. vanish if two are equal.

  So let us study all things that are
  1. multi-linear (Sec 26)
  2. Vanish if two arguments are equal (Sec 27).

Def. $\mathcal{L}^k(V) := \left\{ f: V^k \to \mathbb{R} : \begin{array}{l} v \mapsto f(v_1 \ldots v_{i-1}, \underset{i}{v}, v_{i+1} \ldots v_k) \text{ is} \\ \text{linear in } v, \text{ for all } 1 \le i \le k \end{array} \right\}$

For $k=1$, this is V^*, "the dual of V". In general, these are "k-tensors".

Claim $\mathcal{L}^k(V)$ is a vector space.

Claim If $(a_1 \ldots a_n)$ is a basis for V, and $I = (i_1 \ldots i_k) \in \{1, \ldots n\}^k = \underline{n}^k$ then there is a unique $\phi_I \in \mathcal{L}^k(V)$ s.t. for every $J = (j_1 \ldots j_k) \in \underline{n}^k$

$$(*) \qquad \phi_I(a_{j_1} \ldots a_{j_k}) = \begin{cases} 1 & I = J \\ 0 & \text{otherwise} \end{cases} = \delta_{IJ}$$

These ϕ_I make a basis of $\mathcal{L}^k(V)$ (so $\dim \mathcal{L}^k(V) = n^k$)

steps. 1. An element of $\mathcal{L}^k(V)$ is determined by it's values on seqs. of basis vectors, so if exists, ϕ_I is unique.

2. If $k=1$, $\phi_i(a_j) = \delta_{ij}$ determines $\phi_i \in \mathcal{L}^1(V) = V^*$.

3. Set $\phi_I(v_1 \ldots v_k) = \phi_{i_1}(v_1) \cdot \phi_{i_2}(v_2) \ldots \phi_{i_k}(v_k)$; it is in $\mathcal{L}^k(V)$ and it satisfies $(*)$.

4. The ϕ_I's are lin-indep.

5. The ϕ_I's span $\mathcal{L}^k(V)$.