1617-257 Mon Jan 23, hour 44: Manifolds and their boundaries

January 16, 2017 8:21 AM

$$\int_M d\omega = \int_{\partial M} \omega$$

Hour 44 Handout

- Term test 2 return and discussion today at 14:50.
- Today's reading: Still sections 23 and 24.
- Note that in the "inline" version of HW11, problem 2 of section 20 is misquoted. The "inline" assignments are prepared by a student — an excellent idea, though please include a disclaimer and link to the original, which is at http://drorbn.net/index.php?title=1617-257/Homework_Assignment_11.
- HW12 will be at http://drorbn.net/index.php?title=1617-257/Homework_Assignment_12 by midnight today.

Let $H^k = \{x \in \mathbb{R}^k \colon x_k \geq 0\}$ denote the k-dimensional "upper half space".

Definition 1. A k-manifold of class C^r, possibly with boundary, in $\mathbb{R}^n$ is a subset $M \subset \mathbb{R}^n$ such that each $p \in M$ has an open neighborhood V (in M) such that there is an open $U \subset H^k$ and a C^r homeomorphism $\alpha \colon U \to V$ (a "coordinate patch") whose differential has rank k for every $x \in U$. The boundary ∂M of M is

$$\partial M = \left\{ p \in M \colon \begin{array}{l} \text{for some patch } \alpha,\ p = \alpha(q) \\ \text{with } q \in \partial H^k := \mathbb{R}^{k-1} \times \{0\} \end{array} \right\}.$$

Issues.

1. How do we define "a C^r function on a non-open set in $\mathbb{R}^k$" (such as H^k)?
2. Why isn't $\partial M = M$?
3. Can we prove the theorem below?

Theorem 1. With M as above, ∂M is a $(k-1)$-manifold of class C^r with no boundary.

Definition 2. Let S be an arbitrary subset of $\mathbb{R}^k$. A function $f \colon S \to \mathbb{R}^n$ is said to be of class C^r if for every $p \in S$ we can find an open subset U_p of $\mathbb{R}^k$ containing p, and a C^r function $g_p \colon U_p \to \mathbb{R}^n$, such that $g_p(x) = f(x)$ for every $x \in S \cap U_p$.

Note that this is *not* the definition we mentioned in class last week. However, the following theorem says that this definition and the one from last week are equivalent.

Theorem 2. Let S be an arbitrary subset of $\mathbb{R}^k$. A function $f \colon S \to \mathbb{R}^n$ is of class C^r iff there exists an open subset U of $\mathbb{R}^k$ containing S and a C^r function $g \colon U \to \mathbb{R}^n$ such that $g(x) = f(x)$ for every $x \in S$.

To prove Theorem 2 we need a lemma, which is in fact a much more important theorem (recall first that the support supp(ϕ) of a function ϕ is the closure of the set of points where it is not vanishing):

The Partitions of Unity Lemma. Given a collection $\mathcal{A}$ of open sets in $\mathbb{R}^k$ whose overall union is $A = \bigcup_{U \in \mathcal{A}} U$, there exists a sequence $\{\phi_i\}$ of non-negative compactly-supported C^∞ functions such that:

1. For each i there is some $U \in \mathcal{A}$ such that $\mathrm{supp}(\phi_i) \subset U$.
2. Every $x \in A$ has a neighborhood V such that $\{i \colon V \cap \mathrm{supp}(\phi_i) \neq \emptyset\}$ is finite.
3. $\sum_{i=1}^{\infty} \phi_i = 1$ on A.

Such a sequence $\{\phi_i\}$ is called "a partition of unity subordinate to $\mathcal{A}$".

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Term Test 2. 90 students took the test. The results so far, before appeals and with one test left to mark, are (median underlined):

100 100 100 100 99 97 97 97 96 95 94 93 93 92 90
90 90 90 90 89 88 88 87 85 85 85 84 84 84 83 83
80 79 79 78 77 76 75 75 75 73 73 72 72 <u>72</u> 72 71
71 70 69 68 66 66 65 65 64 63 62 62 62 62 61 59
59 58 58 57 56 55 54 53 53 53 53 53 53 52 49 45
42 39 38 38 36 35 35 31 29 29

The results are similar to what I expected them to be and to the results of the previous term test (though a bit better). Please re-read my comments at http://drorbn.net/index.php?title=1617-257/Term_Test_1.

Appeals. Remember! We try hard yet grading is a difficult process and mistakes always happen — solutions get misread, parts are forgotten, grades are not added up correctly. You must read your exam and make sure that you understand how it was graded. If you disagree with anything, don't hesitate to complain! (Though first consider very carefully the possibility that the mistake is actually yours). Your first stop should be the person who graded the problem in question, and only if you can't agree with him you should appeal to Dror (within a further day or two).

Dror marked problems number 3 and 6 and did the arithmetic and data entry. Jeffrey Im marked the rest.

The deadline to start the appeal process is Monday January 30 at 3PM. Once you've started the process by talking to Dror or to Jeffrey, it ends when a final decision is made, with no deadline.

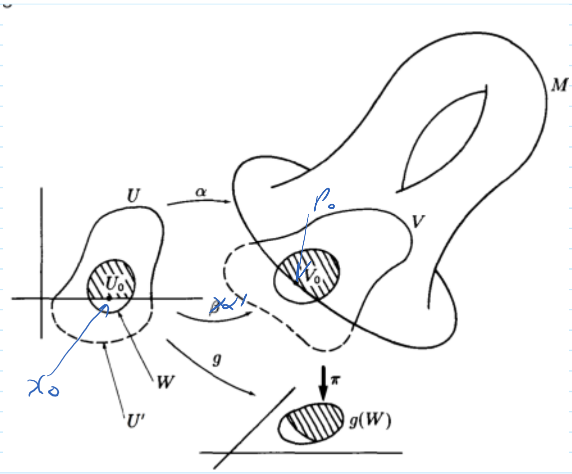
Pf of locality of diffability on S. Let $\mathcal{A} = \{U \colon F|_{S \cap U} \text{ has a diff'ble extension to } U\}$. The $A = \bigcup_{U \in \mathcal{A}} U$ contains S. Find a partition of unity $\{\phi_i\}$ subordinate to $\mathcal{A}$, for each i find $U_i \in \mathcal{A}$ s.t. $\mathrm{supp}\,\phi_i \subset U_i$ and a C^r function $g_i \colon U_i \to \mathbb{R}$ s.t. $g|_{S \cap U_i} = F$. Let h_i be $\phi_i \cdot g_i$ extended to A by 0. Then $\sum h_i$ is a C^r function on A and $h|_S = F$.

Proposition If M^k is a class C^r manifold in $\mathbb{R}^n$ and $\alpha \colon U \subset H^k \to V \subset M^k$ is a coordinate patch, then $\alpha^{-1} \colon V \to U$ is C^r.

Proof

corollary "transition functions" are C^r.

Proof



Corollary "Transition Functions" are C^r.

continue to I2 & I3 !