1617-257 Mon Jan 16, hour 41: k-vols in Rn (general)

January 11, 2017 3:23 PM

TT: Tomorrow 5PM-7PM @ EX 300. Extra OH: Dror Mon Jan 16 5:30-8 BA 6178, Jeff Tue Jan 17 11-2 Huron 215 10th floor. Covers everything except today's material.

Good Luck!!!

Read along: Sec 22.

The volume of the parallelepiped spanned by $x_1, \dots, x_k \in \mathbb{R}^n$ is

$$V(x_1 \dots x_n) = |\det X^T X|^{1/2} \qquad X = (x_1 | \dots | x_k)$$

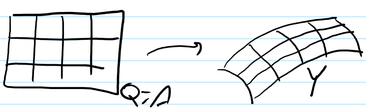
* rotation invariant * "correct" on $\mathbb{R}^k$. on board

Def A "parametrized k-manifold in $\mathbb{R}^n$" is a C^1 map $\alpha : A \longrightarrow \mathbb{R}^n$, where A is open in $\mathbb{R}^k$. We tend to think of $\alpha(A) = Y$ as "the manifold" and of α as "its parametrization".

Def The "volume" of Y is:

$$V(Y) = V(Y, \alpha) := \int_A V(D\alpha) := \int_A |\det (D\alpha)^T D\alpha|^{1/2}$$

Rationalization. Suppose $A = Q$, P a fine partition of Q.

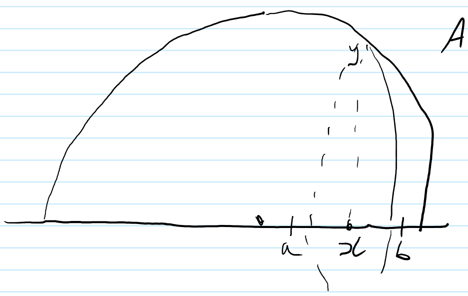
α: 

$$V(Y) \sim \sum_{R \in P} \mathrm{Vol}(\alpha(R)) \sim \sum_{R \in P} V(D\alpha(c) \cdot (c_1 - d_1) e_1 | \dots)$$

$R = \prod [c_i, d_i]$, $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

$$= \sum_{\substack{R \in P \\ R = \prod [c_i, d_i] \\ c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}} \prod (d_i - c_i) \cdot V(D\alpha(c)) \sim \int_Q V(D\alpha)$$

Example. Compute the amount of crust on a slice of a spherical loaf of bread.



$$A = (a, b) \times (0, 2\pi) \subset \mathbb{R}^2_{x,\theta} \qquad \alpha(x, \theta) = \begin{pmatrix} x \\ \sqrt{1-x^2} \cos\theta \\ \sqrt{1-x^2} \sin\theta \end{pmatrix}$$

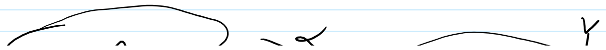
$$D\alpha = \begin{pmatrix} 1 & 0 \\ \frac{-x\cos\theta}{\sqrt{1-x^2}} & -\sqrt{1-x^2}\sin\theta \\ \frac{-x\sin\theta}{\sqrt{1-x^2}} & \sqrt{1-x^2}\cos\theta \end{pmatrix}$$

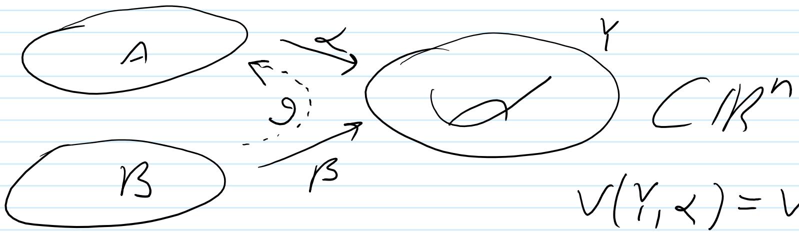
$$(D\alpha)^T (D\alpha) = \begin{pmatrix} 1 + \frac{x^2}{1-x^2} & 0 \\ 0 & 1-x^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1-x^2} & 0 \\ 0 & 1-x^2 \end{pmatrix}$$

$$V(D\alpha) = 1$$

$$\mathrm{Vol} = \int_A 1 = 2\pi (b-a)$$

Thm "The volume of Y is independent of its parametrization"





$$V(Y,\alpha)=V(Y,\beta)$$

precisely, if $g: B \to A$ is a diffeomorphism of open sets in $\mathbb{R}^k$, and $\alpha: A \to \mathbb{R}^n$ is a manifold, set $\beta = \alpha \circ g$ and then $\alpha(A)=\beta(B)=Y$ and $V(Y,\alpha)=V(Y,\beta)$.

done line

Mild generalization: If f is a cont. function on Y, define

$$\int_Y f\,dV := \int_A (f\circ\alpha)\,V(D\alpha)$$

This too is invariant under re-parametrization.