1617-257 Mon Feb 6, hour 50: Alternating Tensors

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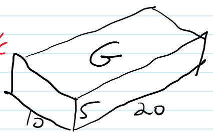
Read Along. Sections 27, 28.

Riddle Along. The Cantor set C is of measure 0. Is it also true for C+C={x+y: x,y\in C}? Is it always true that if B is of measure 0, then so is B+B?

skipped b/c student riddle.



Let V be a v.s. w/ basis $(a_1 \dots a_n)$.

Thm $\forall I \in \underline{n}^k\ \exists!\ \phi_I \in \mathcal{L}^k$ s.t. $\phi_I(a_J) = \delta_{IJ}$; $\{\phi_I\}$ is a basis of $\mathcal{L}^k(V)$.

Definition $\phi \in \mathcal{L}^k(V)$ is "alternating" if $\phi(\dots x \dots y \dots) = -\phi(\dots y \dots x \dots)$.

Claim ϕ is alternating iff $\phi(\dots x \dots x \dots) \equiv 0$ on board

Pf of claim.

Def $A^k(V) = \{\phi \in \mathcal{L}^k(V) : \phi \text{ is alternating}\}$.

Claim $\phi \in A^k(V)$ iff

$\forall \sigma \in S_k,\ \phi(x_1 \dots x_k) = (-1)^\sigma \underbrace{\phi(x_{\sigma 1} \dots x_{\sigma k})}_{\phi^\sigma(x_1 \dots x_k)}$

Def $\underline{n}^k_\wedge := \{(i_1 \dots i_k) \in \underline{n}^k : i_1 < i_2 < \dots < i_k\}$ corrected in class to $\binom{n}{k}$

Thm $\forall I \in \underline{n}^k_\wedge\ \exists!\ \psi_I \in A^k(V)$ s.t.

$\forall J \in \underline{n}^k_\wedge\ \ \psi_I(a_J) = \delta_{IJ}$; $\{\psi_I : I \in \underline{n}^k_\wedge\}$ is a basis of $A^k(V)$ so $\dim A^k(V) = \binom{n}{k}$.

A bit on S_n, then

Aside There is a unique sign assignments $S_n \to \{\pm\}$, denoted $\sigma \mapsto (-1)^\sigma$, s.t.

1. $(-1)^{(ij)} = -1$
2. $(-1)^{\sigma \cdot \tau} = (-1)^\sigma (-1)^\tau$.

NTS: uniqueness, existence.

done line

Pf An alternating tensors is determined by its values on $\{a_J : J \in \underline{n}^k_\wedge\}$ so if ψ_I exists, it is unique.

Existence: $\psi_I(x_1 \dots x_k) := \sum_{\tau \in S_n} (-1)^\tau \phi_I^\tau(x_1 \dots x_k)$

Remains to show: lin indep, span.

Claim In $V = \mathbb{R}^n$ w/ $a_i = e_i$, $\psi_I(x_1 \dots x_k) = \det\begin{pmatrix} x_{1 i_1} & x_{2 i_1} & \dots & x_{k i_1} \\ \vdots & & & \\ x_{1 i_k} & - & - & x_{k i_k} \end{pmatrix}$

In particular, if $I = (1 \dots n)$, $\psi_I(x_1 \dots x_n) = \det(X)$.

Thm $\exists!$ op $\wedge : A^k(V) \times A^l(V) \to A^{k+l}(V)$ s.t.

1. $\wedge$ is associative & bilinear.
2. $\wedge$ is "super-symmetric".
3. $\psi_I = \phi_{i_1} \wedge \phi_{i_2} \wedge \dots \phi_{i_k}$

In addition, if $T : V \to W$, $T^*(f \wedge g) = T^*(f) \wedge T^*(g)$.