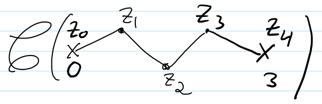
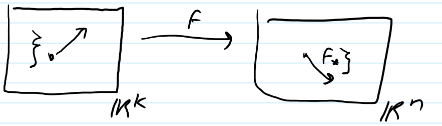
1617-257 Mon Feb 27, hour 56: Tangent Vectors and Forms

February 15, 2017 12:58 PM

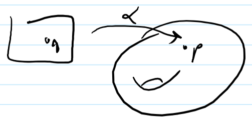
Read Along: Sections 29-30.

Riddle Along.  $= \{(z_0, \dots z_n) \in \mathbb{C}^{n+1} : n=0 \quad z_0 = 0 \quad z_n = n-1 \quad \forall 1\le i\le n \ |z_i - z_{i-1}| = 1\}$



$\xi = (x, v)$

$f_*\xi = (f(x), Df_x v)$



$T_pM := \alpha_* T_q \mathbb{R}^k$

C^r vector fields:

$Y: A \subset \mathbb{R}^n \to \bigcup_{x\in A} T_x\mathbb{R}^n \quad Y(x) \in T_x\mathbb{R}^n \sim \mathbb{R}^n$, so $Y(x) = \sum y_i(x) e_i$

f function $\to Yf$ by $Yf(x) = D_{Y(x)} f = \sum y_i(x) \frac{\partial f}{\partial x_i}$

Y is C^r means $\forall i \ y_i \in C^r \iff [f \in C^{r+1} \Rightarrow Yf \in C^r]$ } proof? on board

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A vector field on a manifold $M \subset \mathbb{R}^n$: $Y = \sum y_i(x) e_i$, s.t. $\forall x \ Y(x) \in T_x M$

Do not push or pull!

k-Tensor fields on open $A \subset \mathbb{R}^n$ $W: A \to$ mess $[= \bigcup_x \mathcal{L}^k(T_x\mathbb{R}^n)]$
s.t. $W(x) \in \mathcal{L}^k(T_x\mathbb{R}^n)$.

So if $\xi_i \in T_x\mathbb{R}^n$, $W(\xi_1 \dots \xi_k) = W(x)(v_1 \dots v_k)$ makes sense.

So if $Y_1 \dots Y_k$ are V.F., $W(Y_1 \dots Y_k)$ is a function.

k-form on an open $A \subset \mathbb{R}^n$
same, but w/ target space $A^k(T_x\mathbb{R}^n)$

In practice, identify all $T_x\mathbb{R}^n$ w/ $\mathbb{R}^n$, and then

$$W = \sum_{I \in \underline{n}^k} a_I(x)\, \phi_I(x) \quad \text{or} \quad W = \sum_{I \in \binom{n}{k}} a_I \psi_I(x)$$

W is C^r means $\forall I \ a_I \in C^r \iff \forall C^r \ Y_1 \dots Y_k$, $W(Y_1 \dots Y_k) \in C^r$

The wedge product — done line

pullbacks

0-forms

Everything also works on manifolds.

$\Omega^k(\mathbb{R}^n) / \Omega^k(M)$ (use C^∞ coeffs)

The "exterior derivative"/"differential" operator $d: \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$

"Right" but hard definition:

$$d\omega(\xi_1 \dots \xi_{k+1}) = \lim_{\epsilon\to 0} \frac{1}{\epsilon^{k+1}} \omega\left(\begin{array}{l}\text{the boundary of the parallelepiped} \\ \text{spanned by } \epsilon\xi_1 \dots \epsilon\xi_{k+1}\end{array}\right)$$

philosofically
"Wrong" but easier definition:

1. On functions, $df(\xi) = D_\xi f = Df_x \cdot v$ $\quad \xi = (x, v)$

   Aside: $x_j : \mathbb{R}^n \to \mathbb{R}$, $dx_j = \phi_j$; hence from this point on, on $\mathbb{R}^n$, dx_I will always replace ϕ_I.

   $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$ $\quad$ [indeed, this works for (x, e_i)]

2. Theorem $\exists! \underbrace{d}_{\text{linear}} : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ s.t.

   1. If f is a 0-form, df is as above.
   2. $\omega \in \Omega^k, \eta \in \Omega^l \Rightarrow d(\omega\wedge\eta) = (d\omega)\wedge\eta + (-1)^k \omega\wedge d\eta$
   3. $d^2 = 0$; more precisely, $d(d\omega) = 0$.