1617-257 Fri Mar 3, hour 58: d and 3D

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Read Along: Sections 29-31. Riddle Along. Your turn!

Today: $\int_M d\omega = \int_{\partial M} \omega$ $\quad \Omega^k(M) :=$ all C^∞ k-forms on M

$d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ by

$$d\omega(\xi_1 \ldots \xi_{k+1}) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^{k+1}} \omega\left(\begin{array}{l}\text{the boundary of the parallelepiped}\\ \text{spanned by } \epsilon\xi_1 \ldots \epsilon\xi_{k+1}\end{array}\right)$$

$\Omega^0(\mathbb{R}^3) \longrightarrow$	$\Omega^1(\mathbb{R}^3)$	$\Omega^2(\mathbb{R}^3)$	$\Omega^3(\mathbb{R}^3)$
f	$a_1(x)\phi_1 + a_2(x)\phi_2 + a_3(x)\phi_3$	$b_1(x)\phi_2\wedge\phi_3 + b_2(x)\phi_3\wedge\phi_1 + b_3(x)\phi_1\wedge\phi_2$	$c\,\phi_1\wedge\phi_2\wedge\phi_3$
functions	v.f.	v.f.	functions

on board

1. On functions, $df(\xi) = D_\xi f = Df_x \cdot v \qquad \xi = (x, v)$

Aside: $x_j : \mathbb{R}^n \to \mathbb{R}$, $dx_j = \phi_j$ hence from this point on, on $\mathbb{R}^n$, dx_I will always replace ϕ_I.

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i \qquad [\text{indeed, this works for } (x, e_i)]$$

2. Theorem $\exists!$ linear operator $d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ s.t.

1. If f is a 0-form, df is as above.
2. $\omega \in \Omega^k, \eta \in \Omega^l \Rightarrow d(\omega\wedge\eta) = (d\omega)\wedge\eta + (-1)^k \omega\wedge d\eta$
3. $d^2 = 0$; more precisely, $d(d\omega) = 0$.

Pf 1-3 imply

$$d(\underbrace{\sum a_I\, dx_I}_{\omega}) = \sum_{j=1}^n \sum_I \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I \overset{\text{loosely}}{=} \sum_{j=1}^n dx_j \wedge \frac{\partial \omega}{\partial x_j}$$

so d is unique, if it exists. As for existence, take the above as the def of d, and verify 1-3.