1617-257 Fri Mar 24, hour 67: Stokes' theorem

February 15, 2017 12:58 PM

Read along: Sec 37-38. Eval rate: 1/93.

practical def of $\int_M \omega$: By chopping in pieces w/ meas-0 exceptions.

Theoretical def of $\int_M \omega$ for compact M: Find a PoU ϕ_i subordinate to "positive charts of M" * $\phi_i \in C^\infty$ * $\operatorname{supp}\phi_i \subset \underset{U_i}{\text{pos. chart}}$ * loc. finite * $\sum \phi_i = 1$

and set $\displaystyle \int_M^{\phi} \omega := \sum_{i\in I} \int_M \phi_i \omega = \sum_{j \in J} \int_M \psi_j \omega = \int_M^{\psi} \omega =: \int_M \omega$

Thm If M^k is compact and oriented and $\omega \in \Omega^{k-1}(M)$, then $\displaystyle\int_M d\omega = \int_{\partial M} \omega$

on board

I should have started with the intuitive proof starting with $d\omega(\xi_1, \dots, \xi_{k+1}) = \lim \frac{1}{\epsilon^{k+1}}(\dots)$

Proof 1. Interior charts $\rightsquigarrow$ $\lambda \in \Omega^{k-1}(\mathbb{R}^k)$ w/ $\operatorname{supp}\lambda \subset \operatorname{int} Q$

$\lambda = \sum \lambda_i \, dx_1 \cdots \widehat{dx_i} \cdots dx_k \qquad d\lambda = \sum (-1)^{i-1} \frac{\partial \lambda_i}{\partial x_i} dx_k$

$$\int_Q d\lambda = \sum_{i=1}^{k} (-1)^{i-1} \int_Q \frac{\partial \lambda_i}{\partial x_i} = 0$$

2. Bndry charts: $\lambda \in \Omega^{k-1}(\underbrace{\mathbb{R}^k_{x_1 \geq 0}}_{H_1^k})$ $\operatorname{supp}\lambda \subset \operatorname{int}_{H_1^k} Q = [0, b_1] \times \overbrace{\prod_{i=2}^{k} [a_i, b_i]}^{\in Q'}$

$$\int_Q d\lambda = \sum_{i=1}^{k} (-1)^{i-1} \int_Q \frac{\partial \lambda_i}{\partial x_i} = \int_Q \frac{\partial \lambda_1}{\partial x_1} = -\int_{x' \in Q'} \lambda_1(0, x') = -\int_{\{0\}\times Q'} \lambda_1 \, dx_2 \wedge \cdots \wedge dx_k = -\int_{\{0\}\times Q'} \lambda = \int_{\partial Q} \lambda$$

3. $\omega = \sum \phi_i \omega$

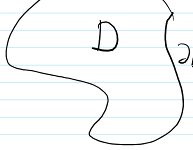
$$\int_{\partial M} \omega = \sum_i \int_{\partial M} \phi_i \omega = \sum_i \int_M d(\phi_i \omega) = \sum_i \int_M d\phi_i \wedge \omega + \phi_i d\omega = \int_M \Big(d\sum \phi_i\Big) \wedge \omega + \Big(\sum \phi_i\Big) \wedge d\omega = \int_M d\omega$$

Example. $M = [0,1]_x$, $\omega = f$

done line

Example

$M = D \subset \mathbb{R}^2 \qquad \omega = P\,dx + Q\,dy \qquad d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$



$\partial D = \operatorname{im}\gamma$

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_{\partial D} P\,dx + Q\,dy = \int_a^b \big(P(\gamma(t))\dot\gamma_1 + Q(\gamma(t))\dot\gamma_2\big) dt$$

"Green's theorem"

$\gamma : [a,b] \to \mathbb{R}^2 = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$

Let $F = \begin{pmatrix} Q \\ -P \end{pmatrix}$; get $\displaystyle\int_D \operatorname{div}(F) = \int_a^b F_1 \dot\gamma_2 - F_2 \dot\gamma_1 = \int_a^b \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \cdot \begin{pmatrix} \dot\gamma_2 \\ -\dot\gamma_1 \end{pmatrix} =$

$= \displaystyle\int F \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot\gamma = \int F \cdot \vec{n}$ the divergence theorem.

Example: Use for $\omega = \frac{1}{2}(x\,dy - y\,dx)$

Example: The Planimeter.