1617-257 Fri Mar 10, hour 61: d and pullbacks

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Read along: Sec 30-32.
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Jeff Tue March 14 11-2 Huron 215 10th floor.

$\phi:\mathbb{R}^n\to\mathbb{R}^m \implies \phi^*:\Omega^k(\mathbb{R}^m)\to\Omega^k(\mathbb{R}^n)$

Claims
1. ϕ^* is linear
2. $(\phi\circ\psi)^*=\psi^*\circ\phi^*$
3. $\phi^*(\omega\wedge\eta)=\phi^*(\omega)\wedge\phi^*(\eta)$

} old

4. $\phi^*(d\omega)=d\phi^*(\omega)$ } owe

Example $\mathbb{R}^2_{r,\theta}\xrightarrow[\binom{r\cos\theta}{r\sin\theta}]{\phi}\mathbb{R}^2_{x,y}$ $\quad \omega=\frac{x\,dy-y\,dx}{x^2+y^2}\in\Omega^2(\mathbb{R}^2_{x,y})$. } only $\phi^*\omega$ computed.

Compute $\underbrace{\phi^*(d\omega)}_{\text{second}}$ & $\underbrace{d\phi^*\omega}_{\text{first}}$

board line

Pf of 4 For functions

$$d(\phi^*f)(\xi)=D_\xi\phi^*f$$
$$\phi^*(df)(\xi)=(df)(\phi_*\xi)=D_{\phi_*\xi}f$$

For general forms:

$$d\left(\phi^*\sum a_I\,dx_I\right)=d\left(\sum\phi^*a_I\ \phi^*(dx_{i_1})\wedge\phi^*(dx_{i_2})\cdots\right)$$
$$=d\left(\sum(\phi^*(a_I)\bigwedge_{\alpha=1}^k\phi^*dx_{i_\alpha}\right)=\sum d\left(\phi^*a_I\bigwedge_{\alpha=1}^k d\phi^*x_{i_\alpha}\right)$$
$$=\sum d\phi^*a_I\wedge\bigwedge_{\alpha=1}^k d\phi^*x_{i_\alpha}$$

$$\phi^*\left(d\sum a_I\,dx_I\right)=\phi^*\left(\sum da_I\wedge\bigwedge_{\alpha=1}^k dx_{i_\alpha}\right)=\sum\phi^*\left(da_I\wedge\bigwedge_{\alpha=1}^k dx_{i_\alpha}\right)$$

Claim If $\phi:\mathbb{R}^n_{x_i}\to\mathbb{R}^n_{y_j}$ & $\omega=f\,dy_{\underline{n}}\in\Omega^{top}(\mathbb{R}^n)$, then $\phi^*(\omega)=\overbrace{\det(D\phi)}^{J_\phi}\cdot\phi^*f\cdot dx_{\underline{n}}$

Example $\mathbb{R}^2_{r,\theta}\xrightarrow[\binom{r\cos\theta}{r\sin\theta}]{\phi}\mathbb{R}^2_{x,y}$ as above; $\omega=dx\wedge dy$; $\phi^*\omega=J_\phi\,dr\wedge d\theta=r\,dr\wedge d\theta$

done line

proof use $\psi_I(x_1\dots x_k)=\det(X_I)$, w/ $X_I=$ rows I of $X=(x_1|\dots|x_k)$ from a while ago:

$\implies dx_{\underline{n}}(v_1\dots v_n)=\det(v_1|\dots|v_n)$ so

$\phi^*(\omega)(e_1\dots e_n)=\omega(\phi_*e_1\dots\phi_*e_n)=f(x)\cdot\det(D\phi)$. This is also the r.h.s.

Claim If $\phi:\mathbb{R}^n_{x_i}\to\mathbb{R}^m_{y_j}$ & $\omega=\sum a_I\,dy_I\in\Omega^k(\mathbb{R}^m)$, then

$$\phi^*(\omega)=\sum_{I\in\binom{m}{k}}\ \sum_{J\in\binom{n}{k}}\phi^*(a_I)\cdot\det\left(D\phi(x)_{J,I}\right)\cdot dx_J$$

The J rows & I col's of $D\phi(x)$.