1617-257 Fri Jan 6, Hour 37: Back to the 257 atmosphere, volumes

December 5, 2016 8:56 AM

(nothing on TT2)

Agenda: Get back to the 257 atmosphere.

Read along: Re-read sections 16-20, esp. 20.

Riddle along: Is there a distance-decreasing function f:[0,1]^2-->R^2 (meaning, d(f(x),f(y))<d(x,y), using the Euclidean metric) such that length(Bd(f([0,1]^2)))>4 ?

<u>Thm</u> given $A \xrightarrow{g} B \xrightarrow{f} \mathbb{R}$, $\int_B f = \int_A (f\circ g)\cdot|\det Dg|$

<u>Corollary.</u> Let $P(v_1\dots v_n)$ be the parallelepiped in $\mathbb{R}^n$ spanned by $v_1\dots v_n$:

$$P(v_1\dots v_n) = \{\sum a_i v_i : 0\leq a_i\leq 1\}$$

Then $\mathrm{vol}(P(v_1\dots v_n)) = |\det(v_1|\dots|v_n)|$

Geometric interp. of dets ?

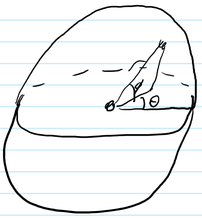
on board

Proof of corollary. Let $g:\mathbb{R}^n_{x_i}\longrightarrow\mathbb{R}^n_{y_i}$ be $g(x_1\dots x_n)=\sum x_i v_i$. Then $Dg=g$ so $|\det Dg| = |\det(v_1|\dots|v_n)|$. Also, $P(v_1\dots v_n)=g([0,1]^n)$ so

$$\mathrm{vol}(P(v_1\dots v_n)) = \int_{P(v_1\dots v_n)} 1 = \int_{[0,1]^n}|\det(v_1\dots v_n)| = |\det(v_1|\dots|v_n)|. \quad \square$$

Exercise. Compute the volume of $B^3=\{x\in\mathbb{R}^3 : |x|\leq 1\}$.

<u>Sol'n</u> use $g(r,\theta,\phi) = (r\cos\theta\cos\phi, r\sin\theta\cos\phi, r\sin\phi)$



$$J=\begin{vmatrix}\cos\theta\cos\phi & -r\sin\theta\cos\phi & -r\cos\theta\sin\phi\\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\phi & 0 & r\cos\phi\end{vmatrix} = \sin^2\phi\cos\phi\cdot r^2 + r\cos^3\phi\cdot r$$

$$= r^3\cos\phi$$

So $V(B^3) = \int_0^1 dr\int_0^{2\pi}d\theta\int_{-\pi/2}^{\pi/2}d\phi\, r^3\cos\phi = \int_0^1 r^3 dr\cdot 4\pi = \frac{4}{3}\pi$

θ: longitude [east-west]

φ: latitude [north-south]

Toronto=(6371, 79, 43)

Def: $h:\mathbb{R}^n\to\mathbb{R}^n$ is an "isometry" if $\forall x,y\ d(h(x),h(y))=d(x,y)$ (Euc)

<u>Thm</u> h is an isometry iff it is of the form

$h(x)=p+Ax$, where $A\in M_{n\times n}$ satisfies $A^TA=I$

Comments 1. Such h is "volume preserving".

done line

2. Such h is a "rotation followed by translation".

[A l.t. is a "rotation" if $A^TA=I$; alt., if it is "orthogonal", meaning it's columns form an "orthonormal basis"]

3. Rotation matrices $O(n):=\{A : A^TA=I\}$ form a "group":

0. $A,B\in O(n) \Rightarrow A\cdot B\in O(n)$.

1. $(AB)C=A(BC)$

2. $\exists I \in O(n)$ s.t. $AI = IA = A$.
3. $\forall A \in O(n)$ $\exists B \in O(n)$ $AB = BA = I$ [use, e.g. $\det(A) = \pm 1$]

<u>proof of thm</u> $\Leftarrow$: easy (though write in full.... ; $\|x\|^2 = x^T \cdot x \dots$)

$\Rightarrow$ steps: 1. WLOG, $h(0) = 0$.

2. h preserves norms.
3. using $\|x+y\|^2 = \dots$, h preserves dot products.
4. set $A = (h(e_1)| \dots |h(e_n))$; then $A \in O(n)$.
5. claim $h(\sum x_i e_i) = \sum x_i h(e_i)$ so $h(x) = Ax$

<u>pf</u> Let $\Delta = h(\sum x_i e_i) - \sum x_i h(e_i)$. Then $\langle \Delta, h(e_j) \rangle = 0$, so $\Delta A e_j = 0$ so $\Delta A = 0$ so $\Delta = 0$.