1617-257 Fri Jan 27, hour 46: Manifolds and boundaries, integration on manifolds

January 16, 2017 8:21 AM

170127-SaddleAndTorus.nb - Wolfram Mathematica 11.0

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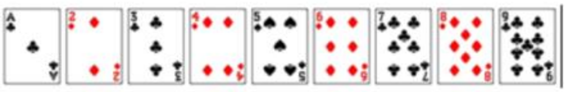
Pensieve header: Plotting a saddle and a torus for January 27.

1617-257 Fri Jan 27, hour 46: Manifolds and boundaries, integration on manifolds.

Read Along. Sections 24-25.

Riddle Along. The game of 15 is played as follows. Two players alternate choosing cards numbered between 1 and 9, with repetitions forbidden, so the game ends at most after 9 moves (or $4\frac{1}{2}$ rounds). The first player to have within their cards a set of precisely 3 cards that add up to 15 wins. Would you rather move first or second?



Anyone looks good (from some angle)

```
ParametricPlot3D[
 {x, y, x^2 - y^2},
 {x, -1, 1}, {y, -1, 1},
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 ViewPoint → {5, 5, 0}, ViewVertical → {0, 0, 1}
]
```
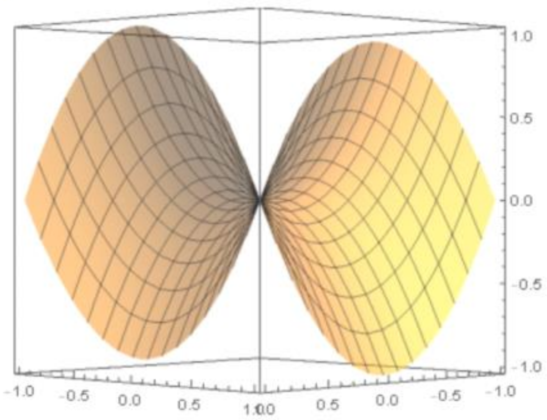
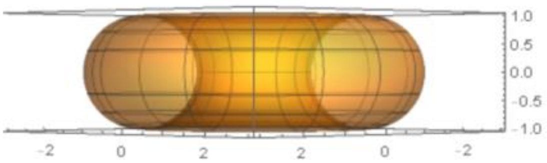
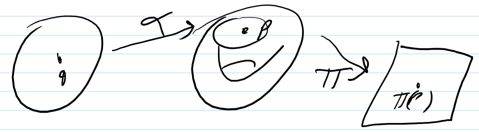



```
ParametricPlot3D[
 {Cos[α] (2 + Cos[β]), Sin[α] (2 + Cos[β]), Sin[β]},
 {α, 0, 2 π}, {β, 0, 2 π},
 PlotStyle -> Opacity[0.5],
 ViewPoint → {5, 5, 0}, ViewVertical → {0, 0, 1}
]
```

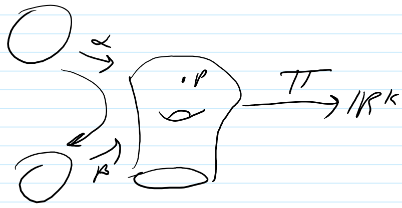


100%

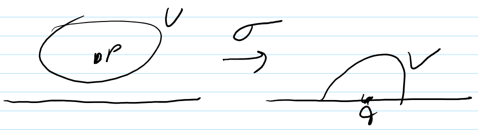


For an patch $\alpha: U^{\subset H^k} \longrightarrow M \subset \mathbb{R}^n$ and any $p = \alpha(q)$, There is a projection $\pi: \mathbb{R}^n \to \mathbb{R}^k$ s.t. $\pi \circ \alpha$ is a diffeo in the vicinity of q; in particular, it has a C^r inverse around $\pi(p)$.

Cor. Trasition functions are C^r:



From last time: No diffeo of open sets in H^k can carry an inner pt. to an edge pt.



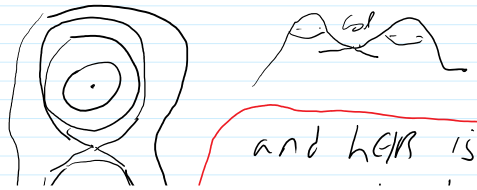
(PF reminder)

Cor ∂M makes sense.

We skip:

∂M is a $(k-1)$-manifold.

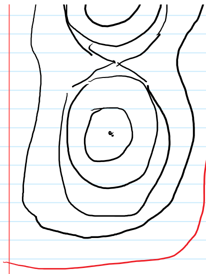
(Easy; patches for M restrict to patches for ∂M)



If $A \subset \mathbb{R}^n$ is open and $f: A \to \mathbb{R}$ is C^r, then for most h $f^{-1}(h)$ is a manifold.

done line.

and $h \in \mathbb{R}$ is such that whenever $p \in f^{-1}(h)$, $df(p)$ has



then for most h $F^{-1}(h)$ is a manifold.

done line.

and $h \in \mathbb{R}$ is such that whenever $p \in F^{-1}(h)$, $dF(p)$ has rank 1, then $F^{-1}(h) = N$ is a manifold, and so are $F^{-1}((-\infty, h]) = M_1$ & $F^{-1}([h, \infty)) = M_2$, and $\partial M_1 = \partial M_2 = N$

Corollary: S^{n-1} is a mfld & $S^{n-1} = \partial D^n$.

Sketch of proof: Implicit function theorem classic!

Integration of Functions on Compact Manifolds:

* Makes sense immediately if function is supported on one patch [that is, if supported on one patch in two ways, both ways give same answer.]
* Can be defined using Po1.
* Indep of choices.

The Partitions of Unity Lemma. Given a collection $\mathcal{A}$ of open sets in $\mathbb{R}^k$ whose overall union is $A = \bigcup_{U \in \mathcal{A}} U$, there exists a sequence $\{\phi_i\}$ of non-negative compactly-supported C^∞ functions such that:

1. For each i there is some $U \in \mathcal{A}$ such that $\text{supp}(\phi_i) \subset U$.
2. Every $x \in A$ has a neighborhood V such that $\{i : V \cap \text{supp}(\phi_i) \neq \emptyset\}$ is finite.
3. $\sum_{i=1}^{\infty} \phi_i = 1$ on A.