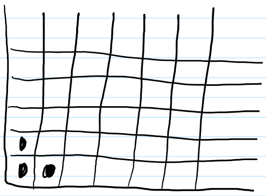
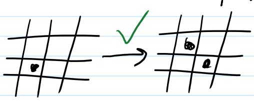
1617-257 Fri Jan 20, hour 43: Manifolds in Rn

January 16, 2017 8:21 AM

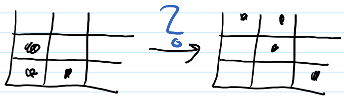
- Still nothing to say about the Term Test.
- Read Along: Sections 23 & 24.
- Riddle Along (Gal's riddle, from 16-475):

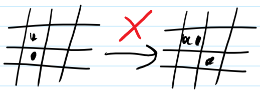


In each move, pick up one piece, remove it, and place one new piece to the right and one new piece above, but only if these squares are unoccupied:



Can you clear the bottom 3?





$$\int_M d\omega = \int_{\partial M} \omega$$

Def A class C^r k-manifold w/o boundary in $\mathbb{R}^n$ is a subset $M \subset \mathbb{R}^n$ s.t. each $p \in M$ has an open neighborhood $V \subset M$ s.t. there is an open $U \subset \mathbb{R}^k$ and a C^r-homeomorphism $\alpha: U \to V$ whose differential has rank k for every $x \in U$.

on board.

Wish to define "mflds w/ bndry" & "bndry ∂M of a mfld M"

Warning: not the same as $Bd\, A$, w/ $A \subset \mathbb{R}^n$!

Premature Definition Let $H^k = \{x \in \mathbb{R}^k : x_k \geq 0\}$

A k-manifold (of class C^r, possibly w/ bndry) in $\mathbb{R}^n$ is $M \subset \mathbb{R}^n$ s.t. each $p \in M$ has an open nbd V s.t. there is an open $U \subset H^k$ & a C^r homeomorphism $\alpha: U \to V$ whose differential has rank k for every $x \in U$.

The bndry ∂M of M is $\partial M = \left\{p \in M : \begin{array}{l}\text{for some patch } \alpha,\\ p = \alpha(q) \text{ w/ } q \in \partial H^k = \mathbb{R}^{k-1} \times \{0\}\end{array}\right\}$

It is a manifold (of class C^r, w/ no bndry) of dim $(k-1)$.

Issues: (I1) What does diffability mean on H^k? What's $d\alpha$ on the edge of H^k?

(I2) Is ∂M well-defined? More precisely, is $\partial M = M$?

(I3) Are we sure ∂M is a manifold?

Dispatch (I1):

Def Let $S \subset \mathbb{R}^k$ & $f: S \to \mathbb{R}^n$. We say that f is a class C^r on S if there is an open $U \supset S$ & a C^r $g: U \to \mathbb{R}^n$ s.t. $g|_S = f$.

Is it necessary? Why not simply use

Easy Fact: If $f: H^k \to \mathbb{R}^n$ is C^r, then $(Df)(p)$ makes sense

Easy Fact: If $F: H^k \to \mathbb{R}^n$ is C^r, then $(DF)(p)$ makes sense for $p \in \partial H^k = \mathbb{R}^{k-1}$; namely, all extentions of F to $U \supset H^k$ have the same differential at p.

Is it necessary? why not simply use the local definition?

Surprisingly hard Fact: "Diffability on S" is a local property. Namely, if $F: S \to \mathbb{R}^n$ is such that every $p \in S$ has a nbd U_p in $\mathbb{R}^k$ s.t. $F|_{S \cap U_p}$ is C^r then F is C^r.

The Partitions of Unity Lemma [Big Theorem] Given a collection $\mathcal{A}$ of open sets in $\mathbb{R}^n$ whose union is A, there exists a sequence $\{\phi_i\}$ of non-negative compactly-supported C^∞ functions s.t.

1. $\forall i$ $\operatorname{supp}\phi_i \subset A$

done line

2. $\forall x \in A$ $\exists$ nbd V of x s.t. $\{i : V \cap \operatorname{supp}(\phi_i) \neq \emptyset\}$ is finite.
3. Each $\operatorname{supp}\phi_i$ is contained in some $V \in \mathcal{A}$.
4. $\sum_{i=1}^{\infty} \phi_i(x) = 1$ on A.

Such a sequence $\{\phi_i\}$ is called "a partition of unity subordinate to $\mathcal{A}$".

Pf of locality of diffability on S Let $\mathcal{A} = \{U : F|_{S \cap U}$ has a diffable extension to $U\}$, the $A = \bigcup_{V \in \mathcal{A}} U$ contains S. Find a partition of unity $\{\phi_i\}$ subordinate to $\mathcal{A}$, for each i find $U_i \in \mathcal{A}$ s.t. $\operatorname{supp}\phi_i \subset U_i$ and a C^r function $g_i : U_i \to \mathbb{R}$ s.t. $g|_{S \cap U_i} = F$. Let h_i be $\phi_i \cdot g_i$ extended to A by 0. Then $\sum h_i$ is a C^r function on A and $h|_S = F$.

continue to I2 & I3 !