1617-257 Fri Jan 13, hour 40: k-vols in Rn (linear)

January 11, 2017 3:23 PM

TT: Tue Jan 17 5PM-7PM @ EX 300. Extra OH: Dror Mon Jan 16 5:30-8 BA 6178, Jeff Tue Jan 17 11-2 Huron 215 10th floor.

Approximate Details:

- Material: Everything from last TT / HW6 until Friday, roughly proportional to time spent + around 20% from older material.
- Roughly choose 4/5, some questions multi-part.
- About 1/3 "prove as in class", 1/3 "solve as in HW", 1/3 "solve fresh".
- How I used to prepare.

Read along: Sec 21.

Riddle Along: Can you fit 4 $a\times b$ rectangles in one $(a+b)^2$ square?
Can you fit 27 $a\times b\times c$ boxes in one $(a+b+c)^3$ cube?
Can you fit 256 $a\times b\times c\times d$ boxes in one $(a+b+c+d)^4$ cube?
Can you fit n^n $\prod_{i=1}^n a_i$ boxes in on $\left(\sum_{i=1}^n a_i\right)^n$ cube?

GS: $v_1' = u_1$ $\qquad v_1 = \pm v_1'/\|v_1'\|$

$v_2' = u_1 - \langle u_1, v_1\rangle v_1$ $\qquad v_2 = \pm v_2'/\|v_2'\|$

$\vdots$

$v_k' = u_k - \sum_{j=1}^{k} \langle u_k, v_j\rangle v_j$ $\qquad v_k = \pm v_k'/\|v_k'\|$

Claim: This works.
pf: formula magic

Thm There's a unique $V : (\mathbb{R}^n)^k \longrightarrow \mathbb{R}_{\geq 0}$ s.t.

1. If $h : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal trans, & $x_i \in \mathbb{R}^n$, then
$$V(h(x_1), \ldots, h(x_k)) = V(x_1 \ldots x_k)$$
2. If $x_i \in \mathbb{R}^k \times \{0\}$, so $x_i = \binom{y_i}{0}$ w/ $y_i \in \mathbb{R}^k$, then
$$V(x_1 \ldots x_k) = |\det(y_1 | \ldots | y_k)|$$

on board

Furthermore, V vanishes iff $\{x_i\}$ are dependent, and
$$V(x_1 \ldots x_k) = |\det \underbrace{X^T \cdot X}_{k\times k}|^{1/2} \qquad \text{w/ } X = (x_1 | \ldots | x_k) \in M_{n\times k}$$
$$\left(\text{so really, } V(x_1 \ldots x_k) = |\det(\langle x_i, x_j\rangle)|^{1/2}\right)$$

Pf 1. Uniqueness: Given $x_1 \ldots x_k$, I'd like to tell you how to compute $V(x_1 \ldots x_k)$ using just 1-2. Find an O.N basis $\{f_i\}_{i=1}^{l}$ of the subspace W generated by $x_1 \ldots x_k$ (so $l \leq k$) and extend it to an O.N. basis $\{f_i\}_{i=1}^n$ of $\mathbb{R}^n$. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $g(e_i) = f_i$. It is O.N., and so is $h := g^{-1}$; $h(f_i) = e_i$. Furthermore, $h(W) = \operatorname{span}\{h(f_i)\}_{i=1}^{l} = \operatorname{span}\{e_i\}_{i=1}^{l} \subset \mathbb{R}^k \times \{0\}$, so $h(x_i) \in \mathbb{R}^k \times 0$, so 2 & 1 determine V.

2. Set $V(x_1 \ldots x_k) = |\det X^T X|^{1/2}$ & prove properties:

   1. $V(h(x_1), \ldots h(x_k)) = V(Ax_1 \ldots Ax_k) = |\det (AX)^T AX|^{1/2} = |\det X^T X|^{1/2}$
   2. If $x_1 \ldots x_k \in \mathbb{R}^k \times \{0\}$, $X = \binom{Y}{0}$ so

$$|\det(X^TX)|^{1/2}=|\det(Y^TY)|^{1/2}=|\det(Y)| \text{ as } Y \text{ is square.}$$

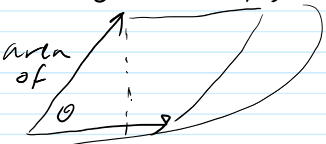
3. $\{x_i\}$ dep. $\Leftrightarrow$ V vanishes:

$\Rightarrow$ x_i dep $\Rightarrow \exists a\neq 0\ Xa=0 \Rightarrow X^TXa=0 \Rightarrow \det(X^TX)=0$.

$\Leftarrow$ $\det(X^TX)=0 \Rightarrow \exists a\neq 0\ X^TXa=0 \Rightarrow a^TX^TXa=0 \Rightarrow Xa=0$.

The $2\subset 3$ case: $V(x,y)=\left|\det\begin{pmatrix}\|x\|^2 & \langle x,y\rangle \\ \langle y,x\rangle & \|y\|^2\end{pmatrix}\right|^{1/2}=\left|\|x\|^2\|y\|^2-\langle x,y\rangle^2\right|^{1/2}$

θ: The angle between x & y.

$=\left|\|x\|^2\|y\|^2-(\|x\|\|y\|\cos\theta)^2\right|^{1/2}=\left|\|x\|\|y\|\sin\theta\right|=$ area of 

done line

$=\left|\|x\times y\|^2\right|^{1/2}=\|x\times y\|$

$$=\left[\sum_{\substack{(i,j):\,2\text{ of }1,2,3\\ i<j}}\begin{vmatrix}x_i & y_i\\ x_j & y_j\end{vmatrix}^2\right]^{1/2}$$

recall

$$x\times y=\begin{vmatrix}e_1 & x_1 & y_1\\ e_2 & x_2 & y_2\\ e_3 & x_3 & y_3\end{vmatrix}=\begin{pmatrix}\begin{vmatrix}x_2 & y_2\\ x_3 & y_3\end{vmatrix}\\ -\begin{vmatrix}x_1 & y_1\\ x_3 & y_3\end{vmatrix}\\ \begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}\end{pmatrix}$$

In general (see book)

$$V(x_1\dots x_n)=\left[\sum_{\substack{(i_1\dots i_k):\,k\text{ of }1\dots n\\ i_1<\dots<i_k}}\det{}^2\begin{pmatrix}x_{1i_1} & \dots & x_{ki_1}\\ x_{1i_2} & & \vdots\\ \vdots & & \\ x_{ki_k} & & x_{ki_k}\end{pmatrix}\right]^{1/2}$$