1617-257 Fri Feb 3, hour 49: Alternating Tensors

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Read Along. Sections 26-27. Riddle Along. Your Turn!

V: v.s. w/ basis $a_1, \dots a_n$

Def $\mathcal{L}^k(V) = \{\text{multi-linear } f: V^k \to \mathbb{R}\}$ ("k-tensors", a v.s.)

For $I \in \underline{n}^k$, $\phi_I \in \mathcal{L}^k(V)$ w/ $\phi_I(a_J) = \phi_I(a_{j_1} \dots a_{j_k}) = \delta_{IJ}$

Done: ϕ_I exists & is unique, $\{\phi_I\}$ is lin. indep.

NTS. $\{\phi_I\}$ spans $\mathcal{L}^k(V)$ [so $\dim \mathcal{L}^k(V) = n^k$]

on board

Easy yet worth noting: There is a map $\otimes: \mathcal{L}^k(V) \times \mathcal{L}^m(V) \to \mathcal{L}^{k+m}(V)$.
It is bilinear, associative, & $\phi_I = \phi_{i_1} \otimes \phi_{i_2} \otimes \dots \phi_{i_k}$

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points, tangent vectors push forward ; functions pull back

Important yet often under-rated: Given a linear $T: V \to W$, there is a "pullback" $T^*: \mathcal{L}^k(W) \to \mathcal{L}^k(V)$. It is

* $*$: linear (meaning - . .)
* $*$ compatible w/ $\otimes$ (meaning, . . .)
* $*$ contravariant: (meaning: . . . .)

Definition A k-tensor $\phi \in \mathcal{L}^k(V)$ is called "alternating" if it changes sign whenever you swap two of its arguments:

$$\phi(\dots x \dots y \dots) = -\phi(\dots y \dots x \dots)$$

done line

Claim ϕ is alternating iff it vanishes whenever two of its arguments are equal:

$$\phi(\dots x \dots x \dots) = 0$$

Pf of claim.

Def $A^k(V) = \{\phi \in \mathcal{L}^k(V) : \phi \text{ is alternating}\}$.

Claim $\phi \in A^k(V)$ iff

$$\forall \sigma \in S_k,\ \phi(x_1 \dots x_k) = (-1)^\sigma \underbrace{\phi(x_{\sigma 1} \dots x_{\sigma k})}_{\phi^\sigma(x_1 \dots x_k)}$$

A bit on S^n, then

Aside There is a unique sign assignments $S_n \to \{\pm\}$, denoted $\sigma \mapsto (-1)^\sigma$, s.t.

1. $(-1)^{(ij)} = -1$
2. $(-1)^{\sigma \cdot \tau} = (-1)^\sigma (-1)^\tau$.

NTS: uniqueness, existence.

If enough prep time: Read sec 27 to verify compatability.

If enough time: Ω^k_n & a basis of $A^k(V)$.