Do not turn this page until instructed.

Math 257 Analysis II

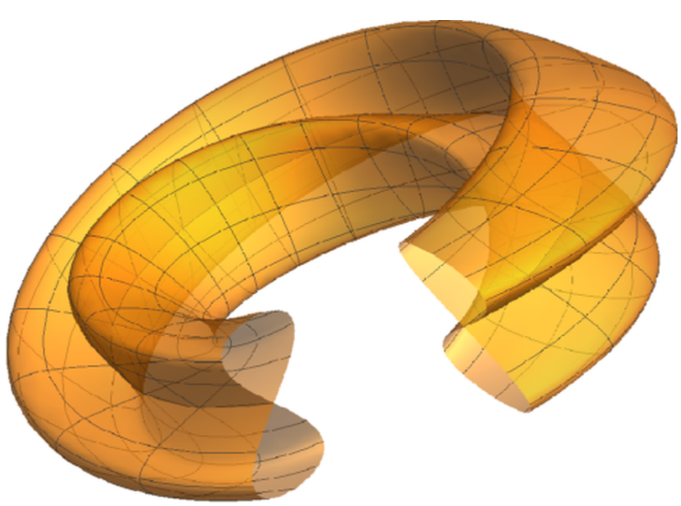
Term Test 3

University of Toronto, March 14, 2017

Solve 4 of the 5 problems on the other side of this page.
Each problem is worth 25 points.
You have an hour and fifty minutes to write this test.

Notes

- No outside material other than stationary is allowed.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and consisting of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.
- Do not write on this examination form! Only what you write in the examination booklets counts towards your grade.
- Indicate clearly which problems you wish to have marked; otherwise an arbitrary subset of the problems you solved will be used.
- In red: post-exam additions/notes.



A Klein bottle with a segment removed

Good Luck!

Solve 4 of the following 5 problems. Each problem is worth 25 points. You have an hour and fifty minutes. **Neatness counts! Language counts!**

Problem 1.

1. State "the chain rule" about the differential of the composition of two functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and $g\colon \mathbb{R}^m \to \mathbb{R}^p$.

2. By appropriately choosing functions $f\colon \mathbb{R} \to \mathbb{R}^2$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$, find the derivative of the function $h(x) = x^x$, for $x > 0$.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Tip. In math exams, "state" means "write the statement of, in full".

Tip. In math exams, "find" means "find and explain how you found".

Problem 2. Let M be a subset of $\mathbb{R}^n$, and let B be the open unit ball in $\mathbb{R}^k$. It is given that two functions $\alpha\colon B \to M$ and $\beta\colon B \to M$ are both homeomorphisms, differentiable of class C^r, and their differentials are of maximal rank at every point of B. Show that the composition $\beta^{-1} \circ \alpha\colon B \to B$ is a C^r diffeomorphism. (You cannot assume that β^{-1} is C^r unless you prove it first).

Tip. In math exams, "show" means "prove".

Problem 3.

1. Write a precise definition of "the pushforward $\phi_*\xi$ of a tangent vector ξ via a C^r map $\phi\colon \mathbb{R}^n \to \mathbb{R}^m$".

2. Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\phi(u, v) = (u^2 - v^2, 2uv)$. Compute $\phi_*(\xi)$, where ξ is the tangent vector given as the pair $\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}\right)$.

Problem 4. Consider the forms $\omega = xydx + 3dy - yzdz$ and $\eta = xdx - yz^2dy + 2xdz$ on $\mathbb{R}^3_{xyz}$. Verify by direct computations that $d(d\omega) = 0$ and that $d(\omega \wedge \eta) = (d\omega) \wedge \eta - \omega \wedge d\eta$.

Problem 5. Explain in detail how the vector-field operator curl arises as an instance of the exterior derivative operator $d\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$, for some k and n.

Good Luck!