Term Test 3 — Problem 4 Marking Key

Problem 4. Consider the forms $\omega = xydx + 3dy - yzdz$ and $\eta = xdx - yz^2dy + 2xdz$ on $\mathbb{R}^3_{xyz}$. Verify by direct computations that $d(d\omega) = 0$ and that $d(\omega \wedge \eta) = (d\omega) \wedge \eta - \omega \wedge d\eta$.

Solution.

$$\begin{aligned} d\omega &= d(xy) \wedge dx + d(3) \wedge dy - d(yz) \wedge dz = (ydx + xdy) \wedge dx + 0 - (zdy + ydz) \wedge dz \\ &= xdy \wedge dx - zdy \wedge dz = -xdx \wedge dy - zdy \wedge dz. \end{aligned} \tag{1}$$

$$d(d\omega) = d(-xdx \wedge dy - zdy \wedge dz) = 0 + 0 = 0. \tag{2}$$

$$\begin{aligned} \omega \wedge \eta &= (xydx + 3dy - yzdz) \wedge (xdx - yz^2dy + 2xdz) \\ &= (-xyyz^2 - 3x)dx \wedge dy + (3 \cdot 2x - yzyz^2)dy \wedge dz + (-yzx - xy2x)dz \wedge dx \\ &= (-xy^2z^2 - 3x)dx \wedge dy + (6x - y^2z^3)dy \wedge dz + (-xyz - 2x^2y)dz \wedge dx \end{aligned} \tag{3}$$

6 terms, rough form

$$\begin{aligned} d(\omega \wedge \eta) &= d((-xy^2z^2 - 3x)dx \wedge dy + (6x - y^2z^3)dy \wedge dz + (-xyz - 2x^2y)dz \wedge dx) \\ &= (-2xy^2z + 6 - xz - 2x^2)dx \wedge dy \wedge dz \end{aligned} \tag{4}$$

4 terms, degree 3

$$\begin{aligned} (d\omega) \wedge \eta &= (-xdx \wedge dy - zdy \wedge dz) \wedge (xdx - yz^2dy + 2xdz) \\ &= (-2x^2 - xz)dx \wedge dy \wedge dz \end{aligned} \tag{5}$$

degree 3

$$\begin{aligned} d\eta &= d(xdx - yz^2dy + 2xdz) \\ &= 2yzdy \wedge dz - 2dz \wedge dx \end{aligned} \tag{6}$$

2 terms, degree 2

$$\begin{aligned} \omega \wedge d\eta &= (xydx + 3dy - yzdz) \wedge (2yzdy \wedge dz - 2dz \wedge dx) \\ &= (2xy^2z - 6)dx \wedge dy \wedge dz \end{aligned} \tag{7}$$

degree 3

$$(4) = (5) - (7) \qquad \text{checks!} \tag{8}$$

Marking Scheme. Look mostly for the items in red, about 4 points for each computation.

Alternative Marking Scheme. 10 for d, 10 for $\wedge$, 5 for arithemetic

(−8). Systematic but wrong differentiation of 1-forms.