Do not turn this page until instructed.

Math 257 Analysis II

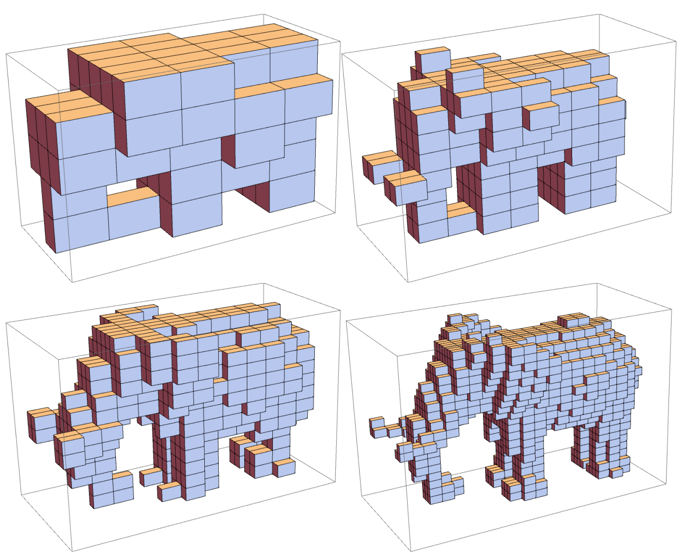
Term Test 2

University of Toronto, January 17, 2017

Solve 4 of the 6 problems on the other side of this page.
Each problem is worth 25 points.
You have an hour and fifty minutes to write this test.

Notes

- No outside material other than stationary is allowed.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and consisting of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.
- Do not write on this examination form! Only what you write in the examination booklets counts towards your grade.
- In red: post-exam additions/notes.
- Indicate clearly which problems you wish to have marked; otherwise an arbitrary subset of the problems you solved will be used.



rectangular approximations of an elephant

Good Luck!

Solve 4 of the following 6 problems. Each problem is worth 25 points. You have an hour and fifty minutes. **Neatness counts! Language counts!**

Problem 1.

1. Define "a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $a \in \mathbb{R}^n$" and "the differential $Df(a)$ of a differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ at $a \in \mathbb{R}^n$".
2. State and prove the "chain rule" for a composition $g \circ f$ of $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and $g\colon \mathbb{R}^m \to \mathbb{R}^p$.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Problem 2. Let $f\colon \mathbb{R}^{k+n} \to \mathbb{R}^n$ be of class C^1; suppose that $f(a) = 0$ and $Df(a)$ has rank n. Prove that if c is a point of $\mathbb{R}^n$ sufficiently close to 0, then the equation $f(x) = c$ has a solution.

Problem 3. (I should have asked to also define "the integral", to emphasize that solutions like "f integrable iff $\mu(D(f)) = 0$" are wrong).

1. Very carefully, define "a bounded function f is integrable on a rectangle Q". Assume that your reader does not know the words / phrases "partition of an interval", "partition of Q", "a rectangle in a partition", "the volume of a rectangle", "lower sum", "upper sum", and whatever remains.
2. Directly from the definitions, prove that constant functions are integrable.

Problem 4.

1. Define: "A set $G \subset \mathbb{R}^2$ is of measure 0".
2. Prove: If $C \subset \mathbb{R}$ is compact, the graph G of a continuous function $f\colon C \to \mathbb{R}$ is of measure 0 in $\mathbb{R}^2$.
3. Prove: The graph of *any* continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is of measure 0 in $\mathbb{R}^2$.

Problem 5. Use the change of variables theorem and Fubini's theorem to compute the two integrals $\int_{\mathbb{R}^2} e^{-(x^2+y^2)/2}dxdy$ and $\int_{\mathbb{R}} e^{-x^2/2}dx$ (in this order!). You need to be clear about how you use these two theorems, yet you do not need to explain why functions that you encounter along the way are integrable.

Problem 6. Let $x_1, \ldots, x_k \in \mathbb{R}^n$ (note that k may be strictly smaller than n here). Prove:

1. Interchanging two of these vectors does not change the volume of the parallelepiped they span.
2. Adding a multiple of one of those vectors to another one does not change the volume of the parallelepiped they span.
3. Multiplying one of these vectors by a scalar $\lambda \in \mathbb{R}$ multiplies the volume of the parallelepiped they span by $|\lambda|$.
4. If these vectors are orthonormal, the volume of the parallelepiped they span is 1.

Tip. For this problem, you may freely use everything stated in class.

Good Luck!