UNIVERSITY OF TORONTO

Faculty of Arts and Sciences

APRIL EXAMINATIONS 2017

Math 257Y1 Analysis II — Final Exam

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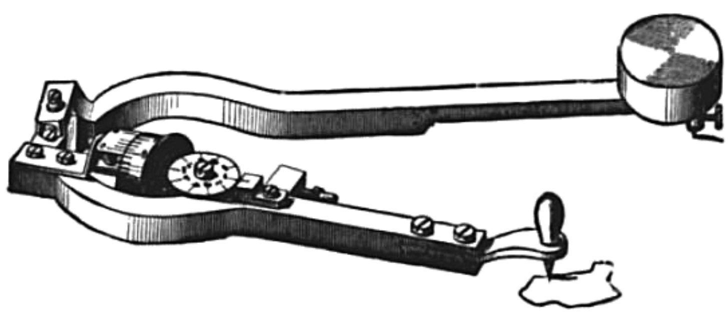
Solve 6 of the following 8 questions. If you solve more than 6 questions indicate very clearly on the examination booklet (not on this form) which are the ones that you want marked, or else arbitrary ones may be excluded. The questions are of equal value of 20 points each, even though they might not be of equal difficulty. [In the computations for the final mark, the results of this exam will be scaled back by a factor of $100/120$]. Should have simply beed "The questions are of equal value".

Duration. You have 3 hours to write this exam.

Allowed Material. None.



Wikipedia: Planimeter: A planimeter (1908) measuring the indicated area by tracing its perimeter.

Good Luck!

Red: Post-factum additions.

Solve 6 of the following 8 problems. Each problem is worth 20 points. You have three hours.

Tip. Neatness, cleanliness, and organization count, here and everywhere else!

Problem 1. Let (X, d) be a metric space, let p_1, p_2, and p_3 be points in X, and let U_1, U_2, and U_3 be open sets in $\mathbb{R}$. Prove that the set

$$A := \left\{x \in X\colon \quad d(x, p_1) \in U_1 \text{ or } \left(d(x, p_2) \in U_2 \text{ and } d(x, p_3) \in U_3\right)\right\}$$

is open in X.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Problem 2. Denote by B_r the open ball of radius r around $0 \in \mathbb{R}^n$ (using, here and below, the Euclidean metric). It is given that a function $f\colon B_1 \to \mathbb{R}^n$ satisfies $f(0) = 0$ and

$$\forall x \neq y \in B_1, \quad |f(y) - f(x) - (y - x)| < 0.1|y - x|.$$

0. Prove that f is continuous.
1. Prove that f is one to one.
2. Prove that f is onto $B_{0.4}$. Namely, for every $z \in B_{0.4}$ there is some $x \in B_1$ such that $z = f(x)$.

Problem 3. Let Q be the rectangle $[a, b] \times [c, d]$ and let f be a real-valued function on Q.

1. Precisely state Fubini's theorem about expressing $\int_Q f$ as an iterated integral.
2. For some such Q, Give an example of a function $g\colon Q \to \mathbb{R}$ for which $\int_{\{x\}\times[c,d]} g$ exists for every $x \in [a, b]$, yet for which $\int_Q g$ does not exist.

Problem 4. Compute the surface area of the two-dimensional sphere $S^2 = \left\{x \in \mathbb{R}^3 \colon |x| = 1\right\}$ in $\mathbb{R}^3$ by writing it as an integral and then evaluating that integral.

Problem 5.

1. (Too open ended) Explain how the cross product $a \times b$ of vectors $a, b \in \mathbb{R}^3$ can be expressed using $\Lambda^1(\mathbb{R}^3)$, $\Lambda^2(\mathbb{R}^3)$, and the operation $\wedge$. 5 points. (1/5) merely defined $a \times b$.

2. Define the operations grad, curl, and div, which act on functions or vector fields on $\mathbb{R}^3$, and output functions or vector fields on $\mathbb{R}^3$. 5 points.

3. (A bit open ended; what does "explain" mean) Explain how grad, curl, and div can be expressed using the "de-Rham complex" 10 points. (4/10) correct diagram, meaningless formulas.
   (−3) all correct, no proofs.

$$\Omega^0(\mathbb{R}^3) \xrightarrow{d} \Omega^1(\mathbb{R}^3) \xrightarrow{d} \Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3).$$

Problem 6. Let $\omega = \dfrac{xdy - ydx}{x^2 + y^2} \in \Omega^1\left(\mathbb{R}^2_{x,y} \setminus \{0\}\right)$, and let $f\colon Q = (0, \infty)_r \times [0, 2\pi]_\theta \to \mathbb{R}^2$ be given by $f(r, \theta) = (r\cos\theta, r\sin\theta)$.

1. Compute $f^*(\omega)$. 5 points.

2. Show that ω is closed. 5 points. (3/5) only showed $f^*\omega$ is closed.

3. Show that $f^*(\omega)$ is exact on Q. 5 points.

4. Show that ω is not exact on $\mathbb{R}^2_{x,y} \setminus \{0\}$. 5 points. (−4) "$\mathbb{R}^2_{x,y} \setminus \{0\}$ isn't $\star$-shaped".

Problem 7. (Open ended problems are very hard to mark!) Write a 1-2 paragraph summary description for each of the (approximately 5) significant ingredients appearing in the formula $\int_M d\omega = \int_{\partial M} \omega$. [You are not required to prove this formula, and it is left to you to choose the significant ingredients that should be described].

Tip. In this problem neatness, cleanliness, and organization especially count, so you may want to write the solution twice; first as a draft, and then in final form. Your summary descriptions must capture the essence of the objects being described, yet they must not be longer than 1-2 paragraph each, or else they will not be read.

"Standard choice": Manifold, boundary, form, d, integration + "orientation".

(−2) No mention of orientations.

Problem 8. Prove the formula $\int_M d\omega = \int_{\partial M} \omega$ by first regarding forms supported within "inner" coordinate charts, then regarding forms supported within "boundary" charts, and then globalizing using a partition of unity. [You may assume without proof that orientation details "work out"].

Tip. In math exams when proving a major theorem, you may assume as known all material that clearly preceded that proof.

(−6) PO1 part missing. (−4) used PO1 w/o specifying which PO1.

Good Luck!