1617-257 Wed Oct 12, Hour 13: Cont. Partials -> Diffability

October 5, 2016 7:15 AM

Chat & pick up HW @ office hours: JI: Fri 11:30–12:30 HU215 10th floor
DBN: Fri 10:30–11:30 Bahen 6178

Today's Agenda: Diffability, deeper.

Read Along: sec 6,7.

Riddle Along: Players A & B alternate putting rectangular Lego pieces of sizes 1x2, 1x3, 1x4 (as they please) on a 19x21 Lego board, w/ no layering allowed. If you no longer have space for a piece, you loose. Whom would you rather be, A or B? What if the overall size was 20x20?

Reminders: $$\frac{|f(a+h)-f(a)-Df(a)\cdot h|}{|h|} \xrightarrow[h\to 0]{} 0$$

$$\sim \quad f(a+h) = f(a) + Df(a)\cdot h + o(h)$$

If f is diffable, $$Df(a) = \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & & \vdots \\ \partial f_m/\partial x_1 & \cdots & \partial f_m/\partial x_n \end{pmatrix}$$

on board

(so in fact, the computation of the differential that we carried out last time was merely tentative)

Thm $f:\mathbb{R}^n \to \mathbb{R}$, $\frac{\partial f}{\partial x_i}$ exist and are cont. near a. Then f is diffable at a. "cont. diffable, class C^1"

Lemma: For any small $h \in \mathbb{R}^n$, $\exists\, q_1 \dots q_n \in U(a,|h|)$ s.t. $$f(a+h)-f(a) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(q_i)\cdot h_i$$

Pf of Thm from lemma: With $B = \left(\frac{\partial f}{\partial x_1}(a) \dots \frac{\partial f}{\partial x_n}(a)\right)$,
$$\frac{f(a+h)-f(a)-B\cdot h}{|h|} = \sum_{i=1}^n \frac{\left[\frac{\partial f}{\partial x_i}(q_i) - \frac{\partial f}{\partial x_i}(a)\right]h_i}{|h|}$$

& rhs $\to 0$.

We need a Lemma^2: If $\phi:[a,b]\to\mathbb{R}$ is cont. on $[a,b]$ and diffable in (a,b), then there is a pt. $c\in(a,b)$ s.t.
$$\phi(b)-\phi(a) = \phi'(c)(b-a)$$

This is the mean value thm (MVT) of 157 0.

Pf of Lemma [given Lemma^2]:

$P_0 = a \quad P_1 = a + h_1 e_1 \quad P_2 = a + h_1 e_1 + h_2 e_2 \quad \dots \quad P_n = a + \sum_{i=1}^{n} h_i e_i = a + h$

Then

$$f(a+h) - f(a) = \sum_{i=1}^{n} f(P_i) - f(P_{i-1}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(q_i) \cdot h_i$$

if $h_i > 0$, using MVT for

$$\phi = f(P_{i-1} + t e_i)$$

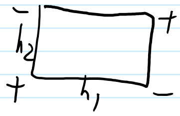
Is "Prove this Theorem" a fair exam question?

<u>Def'n</u> functions of class C^r, C^∞.

<u>Thm</u> If $f: \mathbb{R}^n \to \mathbb{R}$ is of class C^2, then for every $i \& j$, $\partial_i \partial_j f = \partial_j \partial_i f$ [WLOG, $f: \mathbb{R}^2_{x,y} \to \mathbb{R}$]

$a = \binom{x_0}{y_0}$

don't line

Intuition  and why it isn't enough.

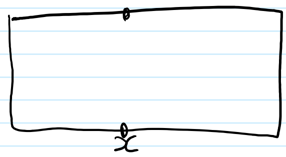
<u>Lemma</u> For any $h > 0$, $\exists p, q \in [x_0, x_0 + h] \times [y_0, y_0 + h]$ s.t.

$$\partial_x \partial_y f(p) = \frac{f(x_0 + h, y_0 + h) - \dots + f(x_0, y_0)}{h^2} = \partial_y \partial_x f(q)$$

Lemma $\Rightarrow$ Thm: easy.

Pf of lemma: Call the numerator above λ, set

$$g(x) = \frac{f(x, y_0 + h) - f(x, y_0)}{h} \quad \text{then } \lambda = \frac{g(x_0 + h) - g(x_0)}{h}$$



then $\exists x_1 \in [x_0, x_0 + h]$ s.t.

$$\lambda = \partial_x g(x_1) = \frac{\partial_x f(x_1, y_0 + h) - \partial_x f(x, y_0)}{h}$$

and then $\exists y_1 \in [y_0, y_0 + h]$ s.t.

$$= \partial_y \partial_x f(x_1, y_1) \implies q = \binom{x_1}{y_1}.$$

$\square$