

1617-257 Wed Nov 30, Hour 33: Improper Integrals (end) and change of variables

October 14, 2016 6:25 AM

Added May 18, 2018: Lax has a very short proof of the change-of-variables formula (see eprints). See also <http://drorbn.net/bbs/show?shot=VanDerVeen-180826-141644.jpg>.

Read Along: Secs 15 and 16-20. Agenda: More improper integrals (bear with me); Jacobians and change-of-variables.

HW9 due, HW10 on web by midnight.

No class Friday! Yes class tomorrow Thu Dec 1 5PM @GB120 (and then riddle hour).

Riddle Along: Show [http://drorbn.net/index.php?title=1617-257/Riddle\\_Repository](http://drorbn.net/index.php?title=1617-257/Riddle_Repository), Nov 28 riddle; what's wrong?

$A \subset \mathbb{R}^n$  open,  $f: A \rightarrow \mathbb{R}$  cont.  $f_{\pm} = \max(\pm f, 0)$   $f = f_+ - f_-$   $|f| = f_+ + f_-$

$f \geq 0 \Rightarrow \int_A f := \sup \left\{ \int_D f : D \subset A \text{ compact and rectifiable} \right\}$   $\int_A f := \int_A f_+ - \int_A f_-$  if all makes sense

Def:  $C_n \setminus A$  " $C_n$  rises to  $A$ " means 1.  $C_n$  compact & rect. 2.  $C_n \cap \text{int}(A)$  3.  $\cup C_n = A$

Thm Given  $C_n \setminus A$ ,  $\int_A f$  exist iff  $\int_{C_n} |f|$  is bndd, and then  $\int_A f = \lim_{n \rightarrow \infty} \int_{C_n} f$ .

Thm If  $A \subset \mathbb{R}^n$  is bndd open &  $f: A \rightarrow \mathbb{R}$  is bndd cont.,

Then  $\int_A f$  exists. If also  $\int_D f$  exists, then  $\int_A f = \int_D f$ .

proof [possibly skip] 1. For  $D \subset A$  compact rectifiable,  $\int_D |f| \leq \left( \sup_D |f| \right) \cdot (\text{Vol of rect containing } D)$ , on board

so  $\int_A f$  exists.

(done on Monday)

2. If  $f \geq 0$ ,  $\int_D f \leq \int_Q f|_A =: \int_A f$ , so  $\int_D f \leq \int_A f$ .

Also, for any partition  $P$  of  $Q$ ,

$$L(f|_A, P) = \sum_{R \in P} m_R(f|_A) V(R) = \sum_{R \in P} m_R(f) V(R) \leq \sum_{R \in P} \int_R f = \int_D f \leq \int_A f$$

where  $D = \cup_{R \in P} R$  is compact rectifiable subset of  $A$

so  $\int_D f \leq \int_A f$ , so  $\int_A f = \int_D f$ .

Now if  $f = f_+ - f_-$  w/  $f_+ = \max(f, 0)$  &  $f_- = \max(-f, 0)$  then  $f_+$  &  $f_-$  are integrable so

$$\int_A f = \int_A f_+ - \int_A f_- = \int_A f_+ - \int_A f_- = \int_A f \quad \square$$

Corollary.  $S$  is bndd &  $f: S \rightarrow \mathbb{R}$  is bndd cont., then  $\int_S f = \int_S f$  (ints)

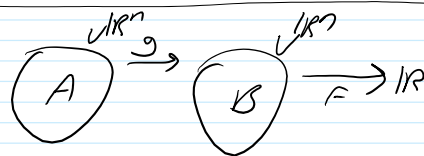
Thm. (skip proof) Let  $A \subset \mathbb{R}^n$  be open,  $f: A \rightarrow \mathbb{R}^n$  cont,  $U_1 \subset U_2 \subset \dots$  open sets s.t.  $\cup U_k = A$ . Then  $\int_A f$  exists iff  $\int_{U_k} |f|$  exist and are bndd, and then,  $\int_A f = \lim_{k \rightarrow \infty} \int_{U_k} f$ .

Examples 1.  $f(x,y) = \frac{1}{\sqrt{xy}}$  on  $A = (1, \infty) \times (1, \infty)$

2. Same  $f$  on  $(0,1) \times (0,1)$ .

Theorem (change of variables, sec 17)

Let  $g: A \rightarrow B$  be a diffeomorphism of open sets in  $\mathbb{R}^n$ . Then  $f$  is integrable on  $B$  iff  $(f \circ g) \cdot |\det Dg|$  is integrable on  $A$ , and in that case,



Diffeomorphism: 1-1 & onto,  $C^1$  with  $C^1$  inverse.

$$\int_B f = \int_A (f \circ g) |\det Dg| = \int_A (f \circ g) J_g$$

"The Jacobian of  $g$ "

Sketch of pf, pass 1 [mention w/o proof that  $|\det L|$  measures volume]

compute  $\int_{\mathbb{R}^2} e^{-(x^2+y^2)/2} dx dy$  in two ways...

1. The most important definite integral in mathematics,  $\int_{-\infty}^{\infty} e^{-x^2/2} dx$

2. use polar coordinates.

(here  $g(r,\theta) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$ ,  $Dg = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$   $J_g = r$ )