# 1617-257 Wed Nov 30, Hour 33: Improper Integrals (end) and change of variables

October 14, 2016 6:25 AM

Added May 18, 2018: Lax has a very short proof of the change-of-variables formula (see eprints). See also http://drorbn.net/bbs/show?shot=VanDerVeen-180826-141644.jpg.

Read Along: Secs 15 and 16-20. Agenda: More improper integrals (bear with me); Jacobians and change-of-variables.
HW9 due, HW10 on web by midnight.
No class Friday! Yes class tomorrow Thu Dec 1 5PM @GB120 (and then riddle hour).
Riddle Along: Show http://drorbn.net/index.php?title=1617-257/Riddle_Repository, Nov 28 riddle; what's wrong?

$A\subset\mathbb{R}^k$ open, $f:A\to\mathbb{R}$ cont. $f_\pm=\max(\pm f,0)$ $f=f_+-f_-$ $|f|=f_++f_-$

$f\geq 0 \Rightarrow \text{ⅈ}\!\!\int_A f := \sup\{\int_D f : D\subset A \text{ compact and rectifiable}\}$ $\quad \text{ⅈ}\!\!\int_A f := \text{ⅈ}\!\!\int_A f_+ - \text{ⅈ}\!\!\int_A f_-$ if all makes sense

Def: $C_n\uparrow A$ "$C_n$ rises to $A$" means 1. $C_n$ cmpct & rect. 2. $C_n\subset \operatorname{int} C_{n+1}$ 3. $\bigcup C_n=A$

Thm Given $C_n\uparrow A$, $\text{ⅈ}\!\!\int_A f$ exist iff $\int_{C_n}|f|$ is bndd, and then $\text{ⅈ}\!\!\int_A f=\lim_{n\to\infty}\int_{C_n} f$.

Thm If $A\subset\mathbb{R}^n$ is bndd open & $f:A\to\mathbb{R}$ is bndd cont., Then $\text{ⅈ}\!\!\int_A f$ exists. If also $\text{ⅇ}\!\!\int_A f$ exists, Then $\text{ⅈ}\!\!\int_A f=\text{ⅇ}\!\!\int_A f$.

on board

proof [possibly skip] 1. For $D\subset A$ compact rectifiable, $\int_D|f|\leq\left(\begin{smallmatrix}\text{bound on}\\ f\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}\text{Vol of}\\ \text{rect containing } A\end{smallmatrix}\right)$, so $\text{ⅈ}\!\!\int_A f$ exists.

(done on Monday)

2. If $f\geq 0$, $\int_D f\leq\int_Q f|_A =: \text{ⅇ}\!\!\int_A f$, so $\text{ⅈ}\!\!\int_A f\leq\text{ⅇ}\!\!\int_A f$.

Also, for any partition $P$ of $Q$,

$$L(f|_A,P)=\sum_{R\in P}m_R(f|_A)v(R)=\sum_{\substack{R\in P\\ R\subset A}}m_R(f)v(R)\leq\sum_{\substack{R\in P\\ R\subset A}}\int_R f=\int_D f\leq\text{ⅈ}\!\!\int_D f$$

where $D=\bigcup_{\substack{R\in P\\ R\subset A}} R$ is cmpct rectifiable subset of $A$

so $\text{ⅇ}\!\!\int_A f\leq\text{ⅈ}\!\!\int_A f$, so $\text{ⅇ}\!\!\int_A f=\text{ⅈ}\!\!\int_A f$.

Now if $f=f_+-f_-$ w/ $f_+=\max(f,0)$ & $f_-=\max(-f,0)$ then $f_+$ & $f_-$ are integrable so

$$\text{ⅇ}\!\!\int_A f=\text{ⅇ}\!\!\int_A f_+-f_-=\text{ⅇ}\!\!\int_A f_+-\text{ⅇ}\!\!\int_A f_-=\text{ⅈ}\!\!\int_A f_+-\text{ⅈ}\!\!\int_A f_-=\text{ⅈ}\!\!\int_A f\qquad\square$$

Corollary. $S$ is bndd & $f:S\to\mathbb{R}$ is bndd cont., Then $\text{ⅇ}\!\!\int_S f=\text{ⅈ}\!\!\int_{\operatorname{int} S} f$

Thm. (skip proof) Let $A\subset\mathbb{R}^n$ be open, $f:A\to\mathbb{R}^n$ cont, $U_1\subset U_2\subset\dots$ open sets s.t. $\bigcup U_k=A$. Then $\text{ⅈ}\!\!\int_A f$ exists iff $\int_{U_k}|f|$ exist and are bndd, and then, $\text{ⅈ}\!\!\int_A f=\lim_{k\to\infty}\text{ⅈ}\!\!\int_{U_k} f$.

Examples 1. $f(x,y)=\frac{1}{x^2y^2}$ on $A=(1,\infty)\times(1,\infty)$

2. Same $f$ on $(0,1)\times(0,1)$.

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Theorem (change of variables, sec 17)

$$\underset{\subset\mathbb{R}^n}{A} \xrightarrow{g} \underset{\subset\mathbb{R}^n}{B} \xrightarrow{f} \mathbb{R}$$

Let $g:A\to B$ be a diffeomorphism of open sets in $\mathbb{R}^n$. Then $f$ is integrable on $B$ iff $(f\circ g)\cdot|\det Dg|$ is integrable on $A$, and in that case, $\int_B f = \int_A (f\circ g)|\det Dg| = \int_A (f\circ g)J_g$

"The Jacobian of $g$"

Diffeomorphism: 1-1 & onto, $C^1$ with $C^1$ inverse.

done line

Sketch of pf, pass 1 [mention w/o proof that $|\det L|$ measures volume]

compute $\int_{\mathbb{R}^2} e^{-(x^2+y^2)/2}$ in two ways...

1. The most important definite integral in mathematics, $\int_{-\infty}^{\infty} e^{-x^2/2}dx$
2. Use polar coordinates.

$\left(\text{here } g(r,\theta)=\begin{pmatrix} r\cos\theta \\ r\sin\theta\end{pmatrix},\ Dg=\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix}\quad J_g=r\right)$