1617-257 Wed Nov 23, Hour 30: Volume Infrastructure

October 14, 2016 6:25 AM

Read Along: Secs 14,15. Agenda: Some volume infrastructure, improper integrals.
HW8 due, HW9 on web by midnight.
Riddle Along: n black/white-hat-wearing prisoners stand in a row; each one sees the hats ahead of them, but not her own or the ones behind. At noon, each one must guess and shout the colour on their head, going from the back forward. If more than is wrong, they are all executed. Could they have devised a strategy in advance to save themselves?

A set is "rectifiable" if 1_S is integrable; the "volume" is then the integral.

Prop. A set is rectifiable iff it is bounded and its boundary is meas-0. on board

Theorem.

1. v(S)>=0.
2. S_1\subset S_2 implies v(S_1)<=v(S_2).
3. If S_i are rectifiable, then v(S_1\union S_2)=V(s_1)+v(S_2)-V(intersection).
4. If S is rectifiable, v(S)=0 iff S is of meas-0.
5. If S is rectifiable and f:S->R is bndd cont, than it is integrable.

Theorem. If C is compact and rectifiable in R^n, and f,g:C->R are cont with f<=g,
Then D={(x,t): x\in C, f(x)<=t<=g(x)} is rectifiable in R^{n+1}, v(D)=\int_C(g-f), and if h is defined on D, then
\int_D h = \int_C(\int_f^g h).

Sketch of proof ... need
Lemma. The graph in R^{n+1} of a cont. function f defined on a compact set C in R^n is of meas-0.

done line

BTW, we don't know yet the volume of a rotated rectangle.

Possibly sec 15 should have been entirely skipped.

~~More...~~ Sec 15: Improper integrals. Goal: $\int_A f$ where $A\subset\mathbb{R}^n$ is open (not bndd) & $f:A\to\mathbb{R}$ is cont. (not bndd)

Examples $\int_{\mathbb{R}}\frac{1}{1+x^2}dx=\pi$ $\quad\int_{(0,\infty)}\frac{1}{x}dx$ diverges.

Several sol'ns: $\fint$ have to ~~then~~ study when they are equal & equal to $\int$

Def 1 If $f\geq 0$, set $\fint_A f=\sup\left(\int_D f:\ \begin{matrix}D\subset A\\ \text{compact}\\ \text{rectifiable}\end{matrix}\right)$

otherwise $\fint_A f:=\int_A f_+-\int_A f_-$, where $f_+=\max\{0,f\}$ $f_-=\max(0,-f)$

(Say that f is $\int$-integrable if this makes sense)

Theorem If $A\subset\mathbb{R}^k$ is open, $f:A\to\mathbb{R}$ cont., C_n a seq. of compact rectifiable sets s.t. $\forall n\ C_n\subset\operatorname{int}C_{n+1}$ & $\bigcup_{n=1}^{\infty}C_n=A$, then $\fint_A f$ exists iff $\int_{C_n}|f|$ is bndd & then $\fint_A f=\lim_{n\to\infty}\int_{C_n}f$

Aside For any A, such a sequence C_n exists?

Pf of aside: $C_n=[-n,n]^k\cap\{x: d(x,A^c)\leq\frac{1}{n}\}$

[Then replace each C_n by a cover by finitely many rectangles still contained in C_{n+1}]

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Pf of thm. If $f\geq 0$ $\fint_A f$ implies $\int_{C_n}f\leq\sup_D\int f=\fint_A f$, so

$\lim_{n\to\infty}\int_{C_n}f$ exists an is $\leq\fint_A f$.

If $\lim_{n\to\infty}\int_{C_n}f$ exist & $D\subset A$ is compact, then $D\subset C_{n_0}$ for some n_0, hence

$\int_D f \le \int_{C_{n_0}} f \le \lim_{n\to\infty} \int_{C_n} f$ so $\fint_A \le \lim_{n\to\infty} \int_{C_n} f$ so $\fint_A f = \lim_{n\to\infty} \int_{C_n} f$.

otherwise $f = f_+ - f_-$ w/ $f_+ \geqslant 0$, $f_- \geqslant 0$, and $|f| = f_+ + f_-$ and then

$\fint_A f$ exists $\Longleftrightarrow$ $\fint_A f_+$ & $\fint_A f_-$ exist $\Longleftrightarrow$ $\lim \int_{C_n} f_+$ & $\lim \int_{C_n} f_-$ exist $\Longleftrightarrow$ $\lim \int_{C_n} |f|$ exist

and in that case, $\lim \int_{C_n} f = \lim \int_{C_n} f_+ - f_- = \lim \int_{C_n} f_+ - \lim \int_{C_n} f_- = \fint_A f_+ - \fint_A f_- = \fint_A f$.