1617-257 Wed Nov 2, Hour 22: Integration, 2

October 14, 2016 6:25 AM

Read Along: Secs 10, 1. Riddle Along: V_L=4V_S. HW6 on web by midnight- cycle changed to Wed->Wed!

Agenda:

$\int_M d\omega = \int_{\partial M} \omega$, basic props of Riemann sums

$f:\mathbb{R}^n \to \mathbb{R}$ bounded $\quad Q = \prod_{j=1}^n [a_j, b_j] \quad \int_Q f$

$\underline{P} = (P_1 \dots P_n) \quad P_j = (a_j = t_{j0} < t_{j1} < t_{j2} < \dots < t_{jk_j} = b_j)$

$R = \prod [c_j, d_j] \in \underline{P} \qquad V(R) = \prod_{j=1}^n (d_j - c_j)$

$m_R(f) = \inf_{x\in R} f(x) \quad L(f, \underline{P}) = \sum_{R\in \underline{P}} m_R(f) V(R) \qquad \underline{\int_Q} f = \sup_{\underline{P}} L(f, \underline{P})$

$M_R(f) = \sup_{x\in R} f(x) \quad U(f, \underline{P}) = \sum_{R\in \underline{P}} M_R(f) V(R) \qquad \overline{\int_Q} f = \inf_{\underline{P}} U(f, \underline{P})$

"f integrable on Q" $\iff \underline{\int_Q} f = \overline{\int_Q} f =: \int_Q f$ $\qquad$ on board.

Def $P' = (a = t_0' < t_1' < \dots < t_{k'}' = b)$ is a refinement of $P = (a = t_0 < t_1 < \dots < t_k = b)$ means $\forall i \; t_i \in \{t_j'\}$. $\underline{P}' = (P_1' \dots P_n')$ is a refinement of $\underline{P} = (P_1, \dots, P_n)$ means $\forall j \; P_j'$ is a refinement of P_j'.

Lemma If $\underline{P}'$ is a refinement of $\underline{P}$, then

$$L(f, \underline{P}') \geq L(f, \underline{P}) \qquad U(f, \underline{P}') \leq U(f, \underline{P})$$

Pf Enough to consider the case where $\underline{P}'$ is obtained from $\underline{P}$ by adding s to to P_{j_0}, between t_{i-1} & t_i. The only difference between $L(f, \underline{P}')$ & $L(f, P)$ is that rectangles of the form

$$R' = \prod_{j=1}^{j_0-1} J_j \times [t_{i-1}, t_i] \times \prod_{j=j_0+1}^n J_j \qquad J_j \in P_j$$

appearing in $L(f, \underline{P}) = \sum_R m_R(f) V(R)$,

Get cut as $R' = R_1 \cup R_2$, where

$$R_1 = \prod_{j=1}^{j_0-1} J_j \times [t_{i-1}, s] \times \prod_{j=j_0+1}^n J_j \quad \text{and} \quad R_2 = \prod_{j=1}^{j_0-1} J_j \times [s, t_i] \times \prod_{j=j_0+1}^n J_j.$$

But $V(R') = V(R_1) + V(R_2)$ & $m_{R_1}(f) \leq m_{R'}(f)$ & $m_{R_2}(f) \leq m_{R'}(f)$

So ...

A similar argument works for $U(f, \underline{P}) \geq U(f, \underline{P}')$.

Lemma For any two $\underline{P}$ & $\underline{P}'$ $\quad L(f, \underline{P}) \leq U(f, \underline{P}')$

Proof: refine both.

Corollary: $\underline{\int_Q} f$ $\overline{\int_Q} f$ make sense & $\underline{\int_Q} f \leq \overline{\int_Q} f$.

Proposition (The Riemann condition) f is integrable iff $\forall \epsilon > 0$ $\exists P$ of Q s.t. $U(f,P) - L(f,P) < \epsilon$.

done line

Example $\int_Q c = c\,\mathrm{Vol}(Q)$

Example If $\lambda(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ then $\int_{[0,1]} \lambda$ does not exist.

Definition. Uniform continuity.

Theorem 2. If f is uniformly continuous over Q then it is integrable.

Theorem 1. Every continuous function on a compact set is uniformly continuous.