1617-257 Wed Nov 16, Hour 27: Fubini

October 14, 2016 6:25 AM

Read Along: Secs 11, 12. HW7 due, HW8 on web by midnight.
Agenda: Fubini.
Riddle Along: Can you cover a diameter 100 disk with 99 (possibly overlapping) 100x1 bands?

<u>Theorem</u> A bndd function $f: Q \to \mathbb{R}$ is integrable iff its disco-set is of measure 0. on board.

<u>Theorem</u> Assume $f: Q \to R$ is integrable.

1. If $f = 0$ almost always (meaning except on a set of meas-0), then $\int_Q f = 0$
   (proof: $[f \neq 0]$ contains no rectangle, hence $[f=0]$ intersects every rectangle, hence $\underline{\int} f \leq 0$ & $\overline{\int} f \geq 0$.)
2. If $f \geq 0$ and the set $\{x \in Q : f(x) > 0\}$ is not meas-0, then $\int_Q f > 0$.
   (proof. Find a pt $a \in [f > 0]$ s.t. f is cont. at a. Find a partition P that has a rectangle R s.t. $f|_R \geq \frac{f(a)}{2}$. Then $L(f,P) \geq \frac{f(a)}{2} v(R)$ so $\underline{\int} \geq$ same)

<u>The Fundamental Thm of Calculus</u> (no proof) Assume f is cont. on $[a,b]$.

1. If $F(x) := \int_a^x f$, then $F'(x)$ exists & $F'(x) = f(x)$
2. If g is s.t. $g' = f$, the $\int_a^b f = g(b) - g(a)$.

(integration and differentiation are opposites; hyped-up telescopic summation; (hyped-up)^2 "your profit in a week is the sum of your daily profits")

Theorem 12.2 (Fubini's theorem). *Let $Q = A \times B$, where A is a rectangle in $\mathbf{R}^k$ and B is a rectangle in $\mathbf{R}^n$. Let $f : Q \to \mathbf{R}$ be a bounded function; write f in the form $f(\mathbf{x}, \mathbf{y})$ for $\mathbf{x} \in A$ and $\mathbf{y} \in B$. For each $\mathbf{x} \in A$, consider the lower and upper integrals*

$$l(x) = \underline{\int}_{\mathbf{y} \in B} f(\mathbf{x}, \mathbf{y}) \quad \text{and} \quad \overline{\int}_{\mathbf{y} \in B} f(\mathbf{x}, \mathbf{y}). = u(x)$$

If f is integrable over Q, then these two functions of $\mathbf{x}$ are integrable over A, and

$$\int_Q f = \int_{\mathbf{x} \in A} \underline{\int}_{\mathbf{y} \in B} f(\mathbf{x}, \mathbf{y}) = \int_{\mathbf{x} \in A} \overline{\int}_{\mathbf{y} \in B} f(\mathbf{x}, \mathbf{y}).$$

Proof. For purposes of this proof, define

$$l(x) \; \underline{I}(\mathbf{x}) = \underline{\int}_{\mathbf{y} \in B} f(\mathbf{x}, \mathbf{y}) \quad \text{and} \quad \overline{I}(\mathbf{x}) = \overline{\int}_{\mathbf{y} \in B} f(\mathbf{x}, \mathbf{y}) = u(x)$$

Example. Let $f(x,y) = x+y$. Then $\int_{[0,1]\times[0,1]} f = \int_0^1\int_0^1 (x+y)\,dx\,dy = 1$

done line

<u>Claim</u> If $P = (P_A, P_B)$ is a partition of Q where P_A is a partition of A and P_B is a partition of B, then

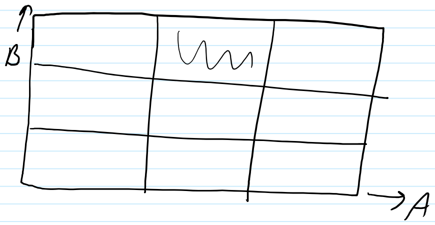
$$L(f,P) \overset{1}{\leq} L(l, P_A) \begin{matrix} \overset{2}{\leq} U(l, P_A) \overset{4}{\leq} \\ \underset{3}{\leq} L(u, P_A) \underset{}{\leq} \end{matrix} \overset{2}{} U(u, P_A) \overset{1}{\leq} U(f,P)$$

claim proves thm: Trivial.

2–5: trivial.

①: $L(\ell, P_A) = \sum_{R\in P_A} v(R)\, m_R(\ell) = \sum_{R\in P_A} v(R) \inf_{x\in R} \sum_{R'\in P_B} v(R') \inf_{y\in R'} f(x,y)$

$\geq \sum_{R\in P_A} v(R) \sum_{R'\in P_B} v(R') \inf_{(x,y)\in R\times R'} f(z) = L(f,P)$



Then run to Rotman L1060!