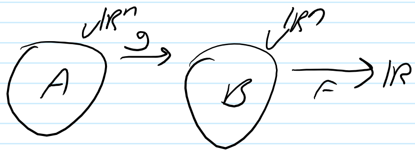
1617-257 Thu Dec 1, Hour 34: Change of variables, Gaussians, volumes of spheres

October 14, 2016 6:25 AM

Added May 18, 2018: Lax has a very short proof of the change-of-variables formula (see eprints). See also http://drorbn.net/bbs/show?shot=VanDerVeen-180826-141644.jpg.

Read Along: Secs 16-20.
Agenda: Tools for the riddle - Jacobians, change-of-variables.
Riddle Along: 1206.nb printout.



__Theorem__ (Change of variables, sec 17)
Let $g: A\to B$ be a diffeomorphism of open sets in $\mathbb{R}^n$. Then f is integrable on B iff $(f\circ g)\cdot|\det Dg|$ is integrable on A, and in that case, $\int_B f = \int_A (f\circ g)|\det Dg| = \int_A (f\circ g)J_g$

Diffeomorphism: 1-1 & onto, C^1 with C^1 inverse.

"The Jacobian of g" on board

Sketch of pf, pass 1 [mention w/o proof that $|\det L|$ measures volume]
compute $G_2 = \int_{\mathbb{R}^2} e^{-(x^2+y^2)/2}$ in two ways....

1. The most important definite integral in mathematics, $G_1=\int_{-\infty}^{\infty} e^{-x^2/2}dx$
2. use polar coordinates.
$\left(\text{here } g(r,\theta)=\begin{pmatrix} r\cos\theta \\ r\sin\theta\end{pmatrix},\ Dg=\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix}\quad J_g=r\right)$

Intuitive proof and the volume of circles. done line

In n-dimensions:

$$(2\pi)^{n/2}=G_n=\int_{\mathbb{R}^n} e^{-\frac{\sum x_i^2}{2}} = \int_0^\infty dr\, e^{-\frac{r^2}{2}}\mathrm{Vol}(S_r^{n-1}) = \int_0^\infty dr\, e^{-r^2/2} r^{n-1}\mathrm{Vol}(S_1^{n-1}) =: M_{n-1} VS_{n-1}$$

$$\text{But } M_{n-1}=\int_0^\infty dr\, \underbrace{e^{-r^2/2}r}_{f'}\cdot \underbrace{r^{n-2}}_{g} = \left[-e^{-r^2/2}r^{n-2}\right]_0^\infty + (n-2)\int_0^\infty e^{-r^2/2}r^{n-3} = (n-2)M_{n-3}$$

$$\text{So } VS_{n-1}=\frac{(2\pi)^{n/2}}{M_{n-1}} = \frac{2\pi(2\pi)^{\frac{n-2}{2}}}{(n-2)M_{n-3}} = \frac{2\pi}{n-2}VS_{n-3} \qquad \text{Also, } VB_n=\frac{1}{n}VS_{n-1}$$

n	1	2	3	4	5
VS_{n-1}	2	2π	4π	$2\pi^2$	
VB_n	2	π	$\frac{4\pi}{3}$	$\frac{1}{2}\pi^2$	