1617-257 Mon Oct 31, Hour 21: Integration

October 14, 2016 6:25 AM

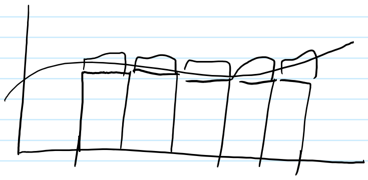
Read Along: Sec 10.
Riddle Along: k kids share a loot of n indivisible candies. The first proposes a split; if not accepted by a strict majority, she leaves and the second proposes, etc. How is the loot split?
TT: Tue Nov 1 5PM-7PM @ BI 131. Extra OH: Jeff Mon 4-7 Huron 215 10th floor, Dror Tue 11-2 BA 6178.
Agenda:

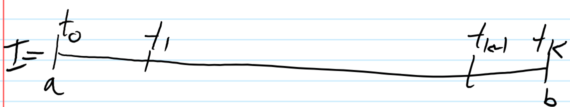
$$\int_M dw = \int_{\partial M} w$$

on board.

$f:\mathbb{R}\to\mathbb{R}$

$$\int_a^b f(x)\,dx = \int_{[a,b]} f$$





$P=(t_0,\dots t_k)$

or $P=\{[t_0,t_1],\dots [t_{k-1},t_k]\}$

$J\in P \iff J$ is one of these

$l(J)=l([c,d])=d-c$

$m_J(f)=\inf\{f(x): x\in J\}$

$M_J(f)=\sup\{f(x): x\in J$

$$L(f,P)=\sum_{J\in P} m_J(f)\,l(J)$$

$U(f,P) = \dots$

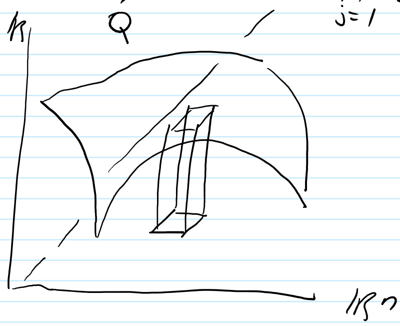
$$\underline{\int_{[a,b]}} f = \sup(L(f,P): P \text{ a partition of } [a,b]$$

$$\overline{\int_{[a,b]}} \quad \dots$$

Def f is integrable on $[a,b]$ if $\underline{\int_{[a,b]}} f = \overline{\int_{[a,b]}} f$ & the common value is the "integral of f on $[a,b]$.

$f:\mathbb{R}^n\to\mathbb{R}$

$$\int_Q f \qquad Q=\prod_{j=1}^n [a_j,b_j]$$



$\underline{P}=(P_1 \dots P_n)$

P_j a partition of $[a_j,b_j]$

$R=\prod[c_j,d_j]\in \underline{P}$ if $\forall j\ [c_j,d_j]\in P_j$

$$v(R)=v\left(\prod[c_j,d_j]\right)=\prod_{j=1}^n (d_j-c_j)$$

$m_R(f)$

$M_R(f)$

$$L(f,\underline{P})=\sum_{R\in \underline{P}} m_R(f)\,v(R)$$

$$\underline{\int_Q} f$$

value is the "integral of f on $[a,b]$, $\int_{[a,b]} f$

done fine

Def $P' = (a = t_0' < t_1' < \dots < t_{k'}' = b)$ is a refinement of $P = (a = t_0 < t_1 < \dots < t_k = b)$ means $\forall i\ t_i \in \{t_j'\}$. $\underline{P}' = (P_1' \dots P_n')$ is a refinement of $\underline{P} = (P_1, \dots, P_n)$ means $\forall j\ P_j'$ is a refinement of P_j'.

Lemma If $\underline{P}'$ is a refinement of $\underline{P}$, then

$$L(f, \underline{P}') \geq L(f, \underline{P}) \qquad U(f, \underline{P}') \leq U(f, \underline{P})$$

Pf Enough to consider the case where $\underline{P}'$ is obtained from $\underline{P}$ by adding s to to P_{j_0}, between t_{i-1} & t_i. The only difference between $L(f, \underline{P}')$ & $L(f, P)$ is that rectangles of the form

$$R' = \prod_{j=1}^{j_0 - 1} J_j \times [t_{i-1}, t_i] \times \prod_{j=j_0+1}^{n} J_j \qquad J_j \in P_j$$

appearing in $L(f, \underline{P}) = \sum_R m_R(f) V(R)$,

Get cut as $R' = R_1 \cup R_2$, where

$$R_1 = \prod_{j=1}^{j_0-1} J_j \times [t_{i-1}, s] \times \prod_{j=j_0+1}^{n} J_j \quad \text{and} \quad R_2 = \prod_{j=1}^{j_0-1} J_j \times [s, t_i] \times \prod_{j=j_0+1}^{n} J_j.$$

But $V(R') = V(R_1) + V(R_2)$ & $m_{R_1}(f) \leq m_{R'}(f)$ & $m_{R_2}(f) \leq m_{R'}(f)$

So....

A similar argument works for $U(f, \underline{P}) \geq U(f, \underline{P}')$.

Lemma For any two $\underline{P}$ & $\underline{P}'$ $L(f, \underline{P}) \leq U(f, \underline{P}')$

Proof: refine both.

Corollary: $\underline{\int_Q} f$ $\overline{\int_Q} f$ make sense & $\underline{\int_Q} f \leq \overline{\int_Q} f$.

Example $\int_Q c = c \operatorname{Vol}(Q)$

Example If $\lambda(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ then $\int_{[0,1]} \lambda$ does not exist. $\square$