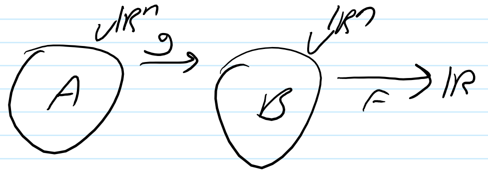
1617-257 Mon Dec 5, Hour 35: C of V, the linear case

October 14, 2016 6:25 AM

Added May 18, 2018: Lax has a very short proof of the change-of-variables formula (see eprints). See also http://drorbn.net/bbs/show?shot=VanDerVeen-180826-141644.jpg.

Wednesday class as usual! (yet no tutorial)
Read Along: your notes. Agenda: as above.
Riddle Along: Infinitely many b/w hat-wearing prisoners watch each other around a round island. At the gong, they all have to guess the colours on their heads, and if more than finitely many get it wrong, the gods of the sea will swallow them all. Could they have devised a strategy for survival in advance?



Theorem (change of variables, sec 17)
Let $g: A \to B$ be a diffeomorphism of open sets in $\mathbb{R}^n$. Then F is integrable on B iff $(F\circ g)\cdot|\det Dg|$ is integrable on A, and in that case, $\int_B F = \int_A (F\circ g)|\det Dg| = \int_A (F\circ g) J_g$

"The Jacobian of g"

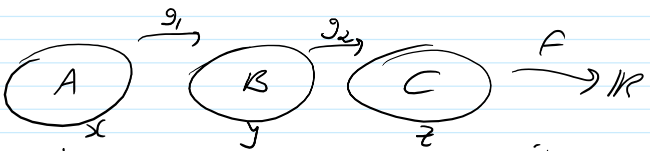
Diffeomorphism: 1-1 & onto, C^1 with C^1 inverse.

on board

Geometry: True for affine linear $g(x) = b + Lx$. Today.
Analysis: Therefore true for any g. Maybe later, or never

* compositions.

Claim



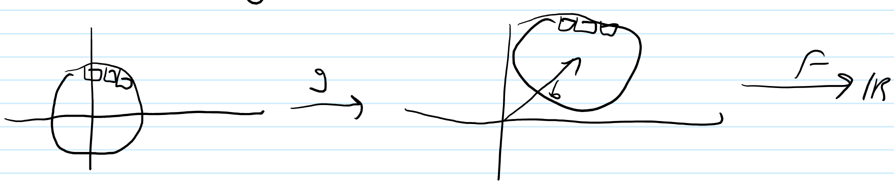
If thm is true for g_1 & for g_2, it is also true for $g_2\circ g_1$.

Proof

$$\int_{z\in C} F = \int_{y\in B} (F\circ g_2)|\det dg_2(y)| = \int_{x\in A} F(g_2(g_1(x)))\cdot|\det dg_2(g_1(x))|\cdot|\det dg_1(x)|$$

$$= \int_{x\in A} F\circ(g_2\circ g_1)\,|\det d(g_2\circ g_1)(x)| \qquad \square$$

* Translations. $g(x) = b + x$



To prove C of V thm if $g = L$ is linear and invertible, it is enough to prove it in the cases of the 3 elementary trans:

* coordinate swaps.

$$\begin{array}{c|c} & \begin{matrix} & i & & j & \end{matrix} \\ \begin{matrix} \\ i \\ j \\ \\ \end{matrix} & \begin{pmatrix} 1 & & & & \\ & 0 & & 1 & \\ & & 1 & & \\ & 1 & & 0 & \\ & & & & 1 \end{pmatrix} \end{array}$$

nothing to show!

* coordinate scalings

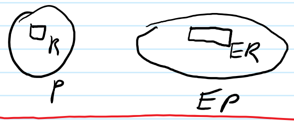
$c \neq 0$

$$\begin{array}{c|c} & \begin{matrix} & & i & \end{matrix} \\ \begin{matrix} \\ \\ i \\ \\ \end{matrix} & \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & c & \\ 0 & & & 1 \end{pmatrix} \end{array}$$

Almost nothing to show!

$$\begin{matrix} \mathbb{R}^n & & \mathbb{R}^n & & \\ \cup & & \cup & & \\ A & \xrightarrow{E^2_{i,c} = E} & B & \xrightarrow{f} & \mathbb{R} \end{matrix}$$

$$\int_B f \overset{?}{=} |c| \int_A f(x_1 \ldots c x_i \ldots x_n)$$



$L(f, EP) = |c| L(f \circ E, P)$ etc.

done line

* shears using Fubini. WLOG, $E_c = \begin{pmatrix} 1 & c & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$.

Need to show:

$$L = \int_{y \in B} f(y_1 \ldots y_n) = \int_{x \in A} \underbrace{f(x_1 + c x_2, x_2, \ldots)}_{= f \circ E_c} = R$$

NTS I: f integrable $\Leftrightarrow$ $f \circ E_c$ integrable II. $L = R$.

Start w/ II: write $x = (x_1, x')$. By Fubini,

$$R \overset{\text{Fub}}{=} \int_{x' \in \mathbb{R}^{n-1}} \left(\int_{x_1 \in \mathbb{R}} f(x_1 + c x_2, x_2, \ldots) \right) \overset{(1)}{=} \int_{x'} \int_{x} f(x_1 \ldots x_n) \overset{\text{Fub}}{=} L.$$

Now I:

$$D(f \circ E_c) = E_c^{-1}(D(f)) = E_{-c}(D(f))$$

Enough, if D is meas-0 in $\mathbb{R}^n$, then so is $E_c(D)$

better, if D is meas-0 in $\mathbb{R}^n$, then so is $L(D)$, where $L: \mathbb{R}^n \to \mathbb{R}^n$ is lin.

enough, given L, $\exists K \in \mathbb{R}$, s.t. for any rectangle R,

$$\text{Vol}(L(R)) \leq K \cdot \text{Vol}(R)$$