1617-257 Fri Nov 25, Hour 31: Improper Integrals

October 14, 2016 6:25 AM

DBN's office hours on Wed 10:30-11:30, from next week.

Read Along: Secs 15. Agenda: Improper integrals, 1st pass.

Riddle Along: Ahmad and Bonita are wearing hats. They know they bear consecutive positive natural numbers, but they don't know what they are.

1. D: A, what number's on your hat? A: I dunno.
2. D: B, what number's on your hat? B: I dunno.
3. D: A, … A: dunno.

….

257. D: What's on your hat? B: I finally know! It's ____.

PS. Why is 6 afraid of 7? on board

~~More…~~ Sec 15.. Improper integrals. Goal: $\int_A f$ where $A\subset\mathbb{R}^n$ is open (not bdd) & $f:A\to\mathbb{R}$ is cont. (not bdd)

Examples $\int_{\mathbb{R}}\frac{1}{1+x^2}dx=\pi$ $\quad\int_{(0,\infty)}\frac{1}{x}dx$ diverges.

Several sol'ns: $\not\int$ $\not\int$ have to ~~their~~ study when they are equal & equal to $\int$

<u>Def 1</u> If $f\geq 0$, set $\not\int_A^E f=\sup_{D\subset A}\left(\int_D f : \begin{matrix}D\subset A\\ \text{compact}\\ \text{rectifiable}\end{matrix}\right)$

otherwise $\not\int_A f := \int_A f_+ - \int_A f_-$, where $f_+=\max\{0,f\}$ $f_-=\max(0,-f)$

(Say that f is E$\not\int$-integrable if this makes sense)

<u>Theorem</u> If $A\subset\mathbb{R}^k$ is open, $f:A\to\mathbb{R}$ cont., C_n a seq. of compact rectifiable sets s.t. $\forall n\, C_n\subset \operatorname{int}C_{n+1}$ & $\bigcup_{n=1}^{\infty}C_n=A$, then $\not\int_A f$ exists iff $\int_{C_n}|f|$ is bdd & then $\not\int_A f=\lim_{n\to\infty}\int_{C_n}f$

In class improv: "$C_n\nearrow A$" "C_n rises to A"

<u>Aside</u> For any A, such a sequence C_n exists?

Pf of aside: $C_n=[-n,n]^k\cap\{x: d(x,A^c)\leq\frac{1}{n}\}$

[Then replace each C_n by a cover by finitely many rectangles still contained in C'_{n+1}]

<u>Pf of thm.</u> If $f\geq 0$ $\not\int_A f$ implies $\int_{C_n}f\leq\sup_D\int f=\not\int_A f$, so

done line

$\lim_{n\to\infty}\int_{C_n}f$ exists an is $\leq\not\int_A f$.

If $\lim_{n\to\infty}\int_{C_n}f$ exist & $D\subset A$ is compact, then $D\subset C_{n_0}$ for some n_0, hence

$\int_D f\leq\int_{C_{n_0}}f\leq\lim_{n\to\infty}\int_{C_n}f$ so $\not\int_A\leq\lim_{n\to\infty}\int_{C_n}f$ so $\not\int_A f=\lim_{n\to\infty}\int_{C_n}f$.

otherwise $f=f_+-f_-$ w/ $f_+\geq 0$, $f_-\geq 0$, and $|f|=f_++f_-$ and then

$\not\int_A f$ exists $\Leftrightarrow$ $\not\int_A f_+$ & $\not\int_A f_-$ exist $\Longleftrightarrow$ $\lim\int_{C_n}f_+$ & $\lim\int_{C_n}f_-$ exist $\Leftrightarrow$ $\lim\int_{C_n}|f|$ exist

and in that case, $\lim\int_{C_n}f=\lim\int_{C_n}f_+-f_-=\lim\int_{C_n}f_+-\lim\int_{C_n}f_-=\not\int_A f_+-\not\int_A f_-=\not\int_A f$.

<u>Thm</u> If A is open in $\mathbb{R}^n$ and f & g are cont.:

1. ${}^{E}\!\!\int_A af+bg = a\,{}^{E}\!\!\int_A f + b\,{}^{E}\!\!\int_A g$ 2. $\int f \leq \int g$ & $|\int f| \leq \int |f|$

3. If $B \subset A$, $\int_B f \leq \int_A f$

4. If A & B are open and f is integrable on A & B,

$$\int_{A\cup B} f = \int_A f + \int_B f - \int_{A\cap B} f$$

Thm If $A \subset \mathbb{R}^n$ is bndd open & $f: A \to \mathbb{R}$ is bndd cont.,

Then ${}^{E}\!\!\int_A f$ exists. If also ${}^{R}\!\!\int_A f$ exists, Then ${}^{E}\!\!\int_A f = {}^{R}\!\!\int_A f$.

proof [possibly skip] 1. For $D \subset A$ compact rectifiable, $\int_D f \leq \left(\substack{\text{bound on}\\ f}\right) \cdot \left(\substack{\text{Vol of}\\ \text{rect containing } A}\right)$,

so ${}^{E}\!\!\int_A f$ exists.

2. If $f \geq 0$, $\int_D f \leq \int_Q f|_A =: {}^{R}\!\!\int_A f$, so ${}^{E}\!\!\int_A f \leq {}^{R}\!\!\int_A f$.

Also, for any partition P of Q,

$$L(f|_A, P) = \sum_{K\in P} m_K(f|_A)v(K) = \sum_{\substack{K\in P\\ K\subset A}} m_K(f)v(K) \leq \sum_{\substack{K\in P\\ K\subset A}} {}^{R}\!\!\int_K f = {}^{R}\!\!\int_D f \leq {}^{E}\!\!\int_D f$$

where $D = \bigcup_{\substack{K\in P\\ K\subset A}} K$ is compact rectifiable subset of A

so ${}^{R}\!\!\int_A f \leq {}^{E}\!\!\int_A f$, so ${}^{R}\!\!\int_A f = {}^{E}\!\!\int_A f$.

Now if $f = f_+ - f_-$ w/ $f_+ = \max(f,0)$ & $f_- = \max(-f,0)$ then f_+ & f_- are integrable so

$${}^{R}\!\!\int_A f = {}^{R}\!\!\int_A f_+ - f_- = {}^{R}\!\!\int_A f_+ - {}^{R}\!\!\int_A f_- = {}^{E}\!\!\int_A f_+ - {}^{E}\!\!\int_A f_- = {}^{E}\!\!\int_A f$$

$\square$