1617-257 Fri Nov 18, Hour 28: Proof of Fubini

October 14, 2016 6:25 AM

Read Along: Secs 12, 13. No riddle!

Agenda - proof of Fubini:

$Q = A \times B \subset \mathbb{R}^{n+m}_{x,y}$, $A \subset \mathbb{R}^n_x$, $B \subset \mathbb{R}^m_y$ rectangles; $f : Q \to \mathbb{R}$ integrable.

$\Rightarrow l(x) = \underline{\int}_B f(x,-)$; $u(x) = \overline{\int}_{y \in B} f(x,y)$ are integrable &

$$\int_Q f = \int_A l = \int_{x \in A} \left(\underline{\int}_{y \in B} f(x,y) \right) = \int_A u = \int_{x \in A} \left(\overline{\int}_{y \in B} f(x,y) \right)$$

on board

Corollaries: 1. If for every $x \in A$ $f(x,y)$ is integrable rel. $y \in B$,

(e.g., if f is cont.) $\int_{A \times B} f = \int_A \left(\int_B f(x,y) \right)$

2. If all integrals exist,

$$\int_B \int_A f(x,y) = \int_Q f = \int_A \int_B f(x,y)$$

3. If $Q = \prod [a_i, b_i]$ & all integrals exist

$$\int_Q f = \int_{x_1 \in [a_1,b_1]} \left(\int_{x_2 \in [a_2,b_2]} \left(\int \dots \int_{x_n \in [a_n,b_n]} f(x_1 \dots x_n) \right) \right) \dots)$$

Lemma If $P = (P_A, P_B)$ is a partition of Q where P_A is a partition of A and P_B is a partition of B, then

$$L(f,P) \overset{①}{\le} L(l, P_A) \overset{②}{\le} U(l, P_A) \overset{④}{\le} U(u, P_A) \overset{⑤}{\le} U(f,P)$$

$$L(l,P_A) \overset{③}{\le} L(u, P_A) \overset{④}{\le} U(u,P_A)$$

I should have found a way to better motivate this lemma

Lemma proves thm: Trivial.

2-5: trivial.

①: $L(l, P_A) = \sum_{R \in P_A} v(R) m_R(l) = \sum_{R \in P_A} v(R) \inf_{x \in R} \sum_{R' \in P_B} v(R') \inf_{y \in R'} f(x,y)$

$$\ge \sum_{R \in P_A} v(R) \sum_{R' \in P_B} v(R') \inf_{(x,y) \in R \times R'} f(z) = L(f,P)$$

done line

Chapter 13 in a nutshell (1. I'm a superstitious coward 2. and chapter is boring)

0. If S is a bounded subset of $\mathbb{R}^n$, define $\int_S f := \int_Q f \cdot 1_S$, where $Q \supset S$ is a rectangle. When defined, it is well-defined.

1. f, g integrable $\Rightarrow \int_S af + bg = a\int_S f + b\int_S g$ (pf: if $a, b \ge 0$ ... then $\int(-f) = -\int f$ ...)

2. f, g integrable, $f \le g \Rightarrow \int_S f \le \int_S g$

3. f integrable $\Rightarrow \left|\int f\right| \leq \int |f|$

4. $T \subset S \Rightarrow \int_T f \leq \int_S f$

5. $$\int_{S_1 \cup S_2} f = \int_{S_1} f + \int_{S_2} f - \int_{S_1 \cap S_2} f$$; if $S_1 \cap S_2$ is meas-0, $\int_{S_1 \cup S_2} f = \int_{S_1} f + \int_{S_2} f$.

6. Read the rest!