Do not turn this page until instructed.

Math 257 Analysis II

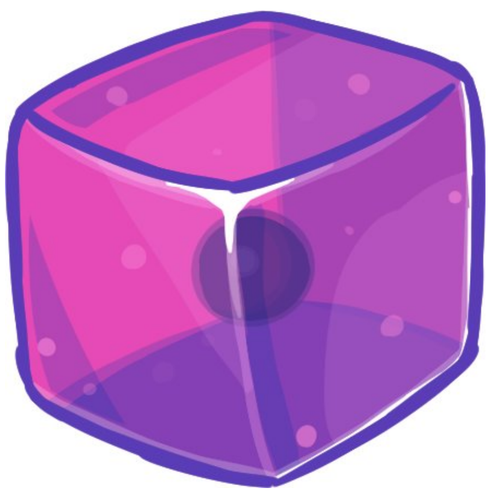
Term Test 1

University of Toronto, November 1, 2016

Solve 4 of the 5 problems on the other side of this page.
Each problem is worth 25 points.
You have an hour and fifty minutes to write this test.

Notes

- No outside material other than stationary is allowed.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and consisting of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.
- In red: post-exam additions/notes.
- Do not write on this examination form! Only what you write in the examination booklets counts towards your grade.



From http://goronic.deviantart.com/art/Jelly-Cube-559481292

Good Luck!

Solve 4 of the following 5 problems. Each problem is worth 25 points. You have an hour and fifty minutes. **Neatness counts! Language counts!**

Problem 1. Let A be a non-empty subset of a metric space (X, d). Show that the distance function to A, defined by $d(x, A) := \inf_{y \in A} d(x, y)$, is a continuous function of x and that $d(x, A) = 0$ iff $x \in \bar{A}$.

Tip. "Iff" means "if and only if", and it always means that there are two things to prove.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Problem 2. Let (X, d) be a metric space.

1. Define "x_0 is a limit point of a subset A of X".
2. Prove that if X is compact and $A \subset X$ is infinite, then A has at least one limit point.

Problem 3.

1. Define "$f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $a \in \mathbb{R}^n$".
2. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $a \in \mathbb{R}^n$, then its differential is uniquely determined.
3. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = |xy|$ is differentiable at $(x, y) = (0, 0)$.

Tip. In math exams, "show" means "prove".

Problem 4.

1. State "the chain rule" about the differential of the composition of two functions $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$.
2. By appropriately choosing functions $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$, find the derivative of the function $h(x) = x^x$.

Tip. In math exams, "state" means "write the statement of, in full".

Tip. In math exams, "find" means "find and explain how you found".

Problem 5. Denote by B_r the open ball of radius r around $0 \in \mathbb{R}^n$ (using, here and below, the Euclidean metric). It is given that a function $f : B_1 \to \mathbb{R}^n$ satisfies $f(0) = 0$ and

$$\forall x, y \in B_1, \quad |f(y) - f(x) - (y - x)| \leq 0.1|y - x|. \qquad \text{(original had "<" by mistake)}$$

(Even better if I had kept it "<" but switched to "$\forall x \neq y$")

1. Prove that f is one to one.
2. Prove that f is onto $B_{0.4}$. Namely, for every $z \in B_{0.4}$ there is some $x \in B_1$ such that $z = f(x)$.

Good Luck!