The Bare Necessities

1. http://drorbn.net/1617-257 2. This will be a tough class. 3. The essence: $\displaystyle\int_M d\omega = \int_{\partial M} \omega$, "Stokes' Theorem".

Like $\int_a^b f' = f|_a^b$, yet: What's M? What's ∂M? What's ω? What's $d\omega$? What's $\int$? Why true? Why care?

Preview: A Bit on Maxwell's Equations

Prerequisites.

- Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.
- Integration by parts: $\int \omega \wedge d\eta = -(-1)^{\deg\omega} \int (d\omega) \wedge \eta$ on domains that have no boundary.
- The Hodge star operator $\star$ which satisfies $\omega \wedge \star\eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.
- The simplesest least action principle: the extremes of $q \mapsto \int_a^b \left(\frac{1}{2} m\dot{q}^2(t) - V(q(t))\right) dt$ occur when $m\ddot{q} = -V'(q(t))$. That is, when $F = ma$.

Table 18–1 Classical Physics

Maxwell's equations	
I. $\nabla \cdot E = \frac{\rho}{\epsilon_0}$	(Flux of E through a closed surface) = (Charge inside)/ϵ_0
II. $\nabla \times E = -\frac{\partial B}{\partial t}$	(Line integral of E around a loop) = $-\frac{d}{dt}$(Flux of B through the loop)
III. $\nabla \cdot B = 0$	(Flux of B through a closed surface) = 0
IV. $c^2 \nabla \times B = \frac{j}{\epsilon_0} + \frac{\partial E}{\partial t}$	c^2(Integral of B around a loop) = (Current through the loop)/ϵ_0 $+\frac{\partial}{\partial t}$(Flux of E through the loop)
[Conservation of charge	
$\nabla \cdot j = -\frac{\partial \rho}{\partial t}$	(Flux of current through a closed surface) = $-\frac{\partial}{\partial t}$(Charge inside)]
Force law	
$F = q(E + v \times B)$	
Law of motion	
$\frac{d}{dt}(p) = F$, where $p = \frac{mv}{\sqrt{1 - v^2/c^2}}$	(Newton's law, with Einstein's modification)
Gravitation	
$F = -G \frac{m_1 m_2}{r^2} e_r$	

The Feynman Lectures on Physics vol. II, page 18-2

The Action Principle. The *4-Vector Potential* is a compactly supported 1-form A on $\mathbb{R}^4$ which extremizes the *action*

$$\boxed{S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} \|dA\|^2 \, dtdxdydz + J \wedge A}$$

where the 3-form J is the *charge-current*.

The Euler-Lagrange Equations in this case are $d \star dA = J$, meaning that there's no hope for a solution unless $dJ = 0$, and that we might as well (think Poincaré's Lemma!) change variables to $F := dA$. We thus get

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

These are the Maxwell equations! Indeed, writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

$dJ = 0 \Longrightarrow$	$\frac{\partial \rho}{\partial t} + \operatorname{div} j = 0$	"conservation of charge"
$dF = 0 \Longrightarrow$	$\operatorname{div} B = 0$	"no magnetic monopoles"
	$\operatorname{curl} E = -\frac{\partial B}{\partial t}$	that's how generators work!
$d * F = J \Longrightarrow$	$\operatorname{div} E = -\rho$	"electrostatics"
	$\operatorname{curl} B = -\frac{\partial E}{\partial t} + j$	that's how electromagnets work!

Exercise. Use the Lorentz metric to fix the sign errors.

Exercise. Use pullbacks along Lorentz transformations to figure out how E and B (and j and ρ) appear to moving observers.

Exercise. With $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ use $S = mc\int_{e_1}^{e_2} (ds + eA)$ to derive Feynman's "law of motion" and "force law".

Quick Review of some Linear Algebra

Let V be a vector space. A set $\mathbf{a}_1, \ldots, \mathbf{a}_m$ of vectors in V is said to **span** V if to each $\mathbf{x}$ in V, there corresponds *at least* one m-tuple of scalars $c_1, \ldots, c_m$ such that

$$\mathbf{x} = c_1\mathbf{a}_1 + \cdots + c_m\mathbf{a}_m.$$

In this case, we say that $\mathbf{x}$ can be written as a **linear combination** of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$.

Theorem 1.2. *Let V be a vector space of dimension m. If W is a linear subspace of V (different from V), then W has dimension less than m. Furthermore, any basis $\mathbf{a}_1, \ldots, \mathbf{a}_k$ for W may be extended to a basis $\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_m$ for V.* □

Theorem 1.6. *If B is the matrix obtained by applying an elementary row operation to A, then*

$$\text{rank } B = \text{rank } A. \quad \square$$

If V has a basis consisting of m vectors, we say that m is the **dimension** of V. We make the convention that the vector space consisting of the zero vector alone has dimension zero.

The set of matrices has, however, an additional operation, called **matrix multiplication.** If A is a matrix of size n by m, and if B is a matrix of size m by p, then the **product** $A \cdot B$ is defined to be the matrix C of size n by p whose general entry c_{ij} is given by the equation

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}.$$

The set $\mathbf{a}_1, \ldots, \mathbf{a}_m$ of vectors is said to be **independent** if to each $\mathbf{x}$ in V there corresponds *at most* one m-tuple of scalars $c_1, \ldots, c_m$ such that

$$\mathbf{x} = c_1\mathbf{a}_1 + \cdots + c_m\mathbf{a}_m.$$

Equivalently, $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ is independent if to the zero vector $\mathbf{0}$ there corresponds only one m-tuple of scalars $d_1, \ldots, d_m$ such that

$$\mathbf{0} = d_1\mathbf{a}_1 + \cdots + d_m\mathbf{a}_m,$$

namely the scalars $d_1 = d_2 = \cdots = d_m = 0$.

Theorem 1.4. *Let V be a vector space with basis $\mathbf{a}_1, \ldots, \mathbf{a}_m$. Let W be a vector space. Given any m vectors $\mathbf{b}_1, \ldots, \mathbf{b}_m$ in W, there is exactly one linear transformation $T : V \to W$ such that, for all i, $T(\mathbf{a}_i) = \mathbf{b}_i$.* □

Theorem 1.1. *Suppose V has a basis consisting of m vectors. Then any set of vectors that spans V has at least m vectors, and any set of vectors of V that is independent has at most m vectors. In particular, any basis for V has exactly m vectors.* □

Theorem 1.5. *For any matrix A, the row rank of A equals the column rank of A.* □

(5) For each k, there is a k by k matrix I_k such that if A is any n by m matrix,

$$I_n \cdot A = A \quad \text{and} \quad A \cdot I_m = A.$$

(1) Exchange rows i_1 and i_2 of A (where $i_1 \neq i_2$).

(2) Replace row i_1 of A by itself plus the scalar c times row i_2 (where $i_1 \neq i_2$).

(3) Multiply row i of A by the non-zero scalar λ.

Theorem 1.3. *If A has size n by m, and B has size m by p, then*

$$|A \cdot B| \leq m|A|\,|B|. \quad \square$$

(1) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.

(2) $A \cdot (B + C) = A \cdot B + A \cdot C$.

(3) $(A + B) \cdot C = A \cdot C + B \cdot C$.

(4) $(cA) \cdot B = c(A \cdot B) = A \cdot (cB)$.

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n),$$
$$c\mathbf{x} = (cx_1, \ldots, cx_n).$$

(1) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.

(2) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$.

(3) $\langle c\mathbf{x}, \mathbf{y} \rangle = c\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, c\mathbf{y} \rangle$.

(4) $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ if $\mathbf{x} \neq \mathbf{0}$.

(1) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

(2) $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$.

(3) There is a unique vector $\mathbf{0}$

(4) $\mathbf{x} + (-1)\mathbf{x} = \mathbf{0}$.

(5) $1\mathbf{x} = \mathbf{x}$.

(6) $c(d\mathbf{x}) = (cd)\mathbf{x}$.

(7) $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$.

(8) $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$.

$$B = \begin{bmatrix} \circledast & * & * & * & * & * \\ 0 & \circledast & * & * & * & * \\ 0 & 0 & 0 & \circledast & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(1) $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$.

(2) $T(c\mathbf{x}) = cT(\mathbf{x})$.

$$C = \begin{bmatrix} 1 & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \cdots + x_n y_n.$$

(1) $(A^{tr})^{tr} = A$.

(2) $(A + B)^{tr} = A^{tr} + B^{tr}$.

(3) $(A \cdot C)^{tr} = C^{tr} \cdot A^{tr}$.

(4) rank A^{tr} = rank A.

(1) $\|\mathbf{x}\| > 0$ if $\mathbf{x} \neq \mathbf{0}$.

(2) $\|c\mathbf{x}\| = |c|\,\|\mathbf{x}\|$.

(3) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

(3′) $\|\mathbf{x} - \mathbf{y}\| \geq \|\mathbf{x}\| - \|\mathbf{y}\|$.

$|\mathbf{x}| = \max\{|x_1|, \ldots, |x_n|\}$. $T(\mathbf{x}) = A \cdot \mathbf{x}$.

standard basis for $\mathbf{R}^n$:

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0),$$
$$\mathbf{e}_2 = (0, 1, 0, \ldots, 0),$$
$$\cdots$$
$$\mathbf{e}_n = (0, 0, 0, \ldots, 1).$$

$|\mathbf{x}| \leq \|\mathbf{x}\| \leq \sqrt{n}|\mathbf{x}|$. $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{y} \rangle^{1/2}$.