# 16-475 Tue Jan 26, hours 7-8: Formulate an Equivalent Problem

January 26, 2016 11:16 AM

My Favourite "Formulate a different problem":

The game of 15.

6 1 8
7 5 3
2 9 4

The Sicherman dice: Can you write positive integers on the side of two blank 6-sided dice so that if thrown, the probability distribution for the sum would be the same as if it had been the ordinary pair of dice, marked (1,2,3,4,5,6) and (1,2,3,4,5,6)?

Soln: $x+x^2+x^3+x^4+x^5+x^6 = x(x+1)(x^2+x+1)(x^2-x+1)$

approx done 9 inc.

and $x(x+1)(x^2+x+1) = x+2x^2+2x^3+x^4$

$x(x+1)(x^2+x+1)(x^2-x+1) = x+x^3+x^4+x^5+x^6+x^8$

Need to find $f_{1,2}$ s.t. 1. $f_i = \sum_{j=1}^{6} x^{k_{ij}}$

$f_i(1)=6$, $f_i$ has pos int coeffs.

2. $f_i \neq x+x^2+\dots+x^6$

# Handout for January 26, "Formulate an Equivalent Problem"

**Class Photo.** Please send me an email identifying yourself in the class photo page. Also let me know if you agree that your name will appear on that page, which is publically accessible. My email remains drorbn@math.toronto.edu.

**Reading.** Section 1.3 of Larson's textbook.

**Next Quiz.** On Thursday January 28, mostly problems from Larson's Section 1.3.

**Problem 1** (Larson's 1.3.1). Find a general formula for the $n$th derivative of $f(x) = 1/(1-x^2)$. Also: integrate

**Problem 2** (Larson's 1.3.2). Find all solutions of $x^4 + x^3 + x^2 + x + 1 = 0$.

**Problem 3** (Larson's 1.3.3). $P$ is a point inside a given triangle $ABC$, and $D$, $E$, and $F$ are the points closest to $P$ on $BC$, $CA$, and $AB$ respectively. Find all $P$ for which

Sol'n:

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$$

is minimal.

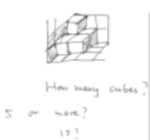
**Problem 5** (Larson's 1.3.5). On a circle $n$ different points are selected and the chords joining them in pairs are drawn. Assuming no three of these chords pass through the same point, how many intersetion points will there be (inside the circle)?

**Problem 6** (Larson's 1.3.6). Given a positive integer $n$, find the number of quadruples of integers $(a, b, c, d)$ such that $0 \leq a \leq b \leq c \leq d \leq n$.

**Problem 7** (Larson's 1.3.7). The number 5 can be expressed as a sum of 3 natural numbers, taking order into account, in 6 ways: $5 = 1+1+3 = 1+2+2 = 1+3+1 = 2+1+2 = 2+2+1 = 3+1+1$. Let $k \leq n$ be natural numbers. In how many ways can $n$ be written as a sum of $k$ natural numbers, minding the order?

**Problem 8.** Same as the previous question, but with "natural numbers" replaced with "non-negative integers".


01.png



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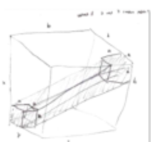
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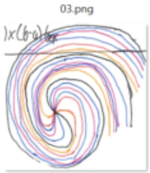
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## Problems

$y^7+7y^6+19y^5+25y^4+15y^3+11y^2 4 \ldots +$

**1.3.8.** Show that $x^7 - 2x^5 + 10x^2 - 1$ has no root greater than 1. (Hint: Since it is generally easier to show that an equation has no *positive* root, we are prompted to consider the equivalent problem obtained by making the algebraic substitution $x = y + 1$.)

not even tight.

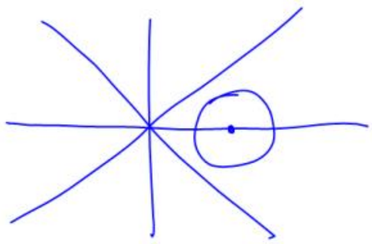
**1.3.9.** The number 3 can be expressed as a sum of one or more positive integers, taking order into account, in four ways, namely, as 3, $1 + 2$, $2 + 1$, and $1 + 1 + 1$. Show that any positive integer $n$ can be so expressed in $2^{n-1}$ ways.

**1.3.10.** In how many ways can 10 be expressed as a sum of 5 *nonnegative* integers, when order is taken into account? (Hint: Find an equivalent problem in which the phrase "5 nonnegative integers" is replaced by "5 positive integers".)

**1.3.11.** For what values of $a$ does the system of equations

$$x^2 = y^2,$$
$$(x-a)^2 + y^2 = 1$$

have exactly zero, one, two, three, four solutions, respectively? (Hint: Translate the problem into an equivalent geometry problem.)

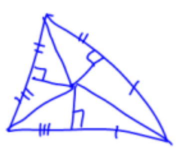


**1.3.12.** Given $n$ objects arranged in a row. A subset of these objects is called *unfriendly* if no two of its elements are consecutive. Show that the number of unfriendly subsets each having $k$ elements is $\binom{n-k+1}{k}$. (Hint: Adopt an idea similar to that used in 1.3.6.)

**1.3.13.** Let $a(n)$ be the number of representations of the positive integer $n$ as a sum of 1's and 2's taking order into account. Let $b(n)$ be the number of representations of $n$ as a sum of integers greater than 1, again taking order into account and counting the summand $n$. The table below shows that $a(4) = 5$ and $b(6) = 5$:

| $a$-sums | $b$-sums |
|---|---|
| $1+1+2$ | $4+2$ |
| $1+2+1$ | $3+3$ |
| $2+1+1$ | $2+4$ |
| $2+2$ | $2+2+2$ |
| $1+1+1+1$ | $6$ |

(a) Show that $a(n) = b(n+2)$ for each $n$, by describing a one-to-one correspondence between the $a$-sums and $b$-sums.
(b) Show that $a(1) = 1$, $a(2) = 2$, and for $n > 2$, $a(n) = a(n-1) + a(n-2)$.



$(a+b+c)r = P_1 a = P_2 b = P_3 c$

$\frac{r}{P_1} = \frac{a}{a+b+c}$

**1.3.14.** By finding the area of a triangle in two different ways, prove that if $p_1, p_2, p_3$ are the altitudes of a triangle and $r$ is the radius of its inscribed circle, then $1/p_1 + 1/p_2 + 1/p_3 = 1/r$.

**1.3.15.** Use a counting argument to prove that for integers $r, n$, $0 < r \leqslant n$,

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}.$$