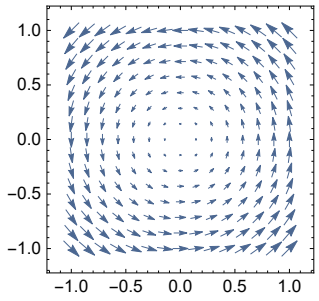
Pensieve header: Drawing phase portraits and drawing the Willerton Fish on April 4, 2016.

From http://www.math.toronto.edu/drorbn/classes/16-1750-ShamelessMathematica/About.html: **Possible Topics** (in no particular order). Whatever you may suggest, and the ~~Fibonacci numbers~~; ~~the Jones polynomial~~; ~~a more efficient Jones-algorithm~~; ~~a riddle on spheres~~; ~~Khovanov homology~~; Γ-calculus; the Hopf fibration; Hilbert's 13th problem; ~~non-commutative Gaussian elimination~~; free Lie algebras; the Baker-Campbell-Hausdorff formula; wacky numbers; ~~an order 4 torus~~; ~~the Schwarz Lantern~~; knot colourings; the Temperley-Lieb pairing; the dodecahedral link; sound experiments; barycentric subdivisions; ~~a Peano curve;~~ braid closures and Vogel's algorithm; ~~the insolubility of the quintic~~; ~~phase portraits~~; ~~the Mamdelbrot set~~; shadows of the Cantor Aerogel; quilt plots; ~~some image transformations~~; De Bruijn graphs; ~~the Riemann series theorem~~; finite type invariants and the Willerton fish.

Linear Phase Portraits

```
VectorPlot[{-y, x}, {x, -1, 1}, {y, -1, 1}]
```



$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{\mathsf{T}}$$

```
{{1, 2}, {3, 4}}
```

```
?? ParametricPlot
```

ParametricPlot[{f_x, f_y}, {u, u_{min}, u_{max}}] generates a parametric plot of a curve with x and y coordinates f_x and f_y as a function of u.

ParametricPlot[{{f_x, f_y}, {g_x, g_y}, …}, {u, u_{min}, u_{max}}] plots several parametric curves.

ParametricPlot[{f_x, f_y}, {u, u_{min}, u_{max}}, {v, v_{min}, v_{max}}] plots a parametric region.

ParametricPlot[{{f_x, f_y}, {g_x, g_y}, …}, {u, u_{min}, u_{max}}, {v, v_{min}, v_{max}}] plots several parametric regions.

ParametricPlot[…, {u, v} ∈ reg] takes parameters {u, v} to be in the geometric region reg. ≫

```
Attributes[ParametricPlot] = {HoldAll, Protected, ReadProtected}
```

```
Options[ParametricPlot] =
 {AlignmentPoint → Center, AspectRatio → Automatic, Axes → True, AxesLabel → None,
  AxesOrigin → Automatic, AxesStyle → {}, Background → None, BaselinePosition → Automatic,
  BaseStyle → {}, BoundaryStyle → Automatic, ColorFunction → Automatic,
  ColorFunctionScaling → True, ColorOutput → Automatic, ContentSelectable → Automatic,
  CoordinatesToolOptions → Automatic, DisplayFunction :→ $DisplayFunction,
  Epilog → {}, Evaluated → Automatic, EvaluationMonitor → None, Exclusions → Automatic,
  ExclusionsStyle → None, FormatType :→ TraditionalForm, Frame → Automatic,
  FrameLabel → None, FrameStyle → {}, FrameTicks → Automatic, FrameTicksStyle → {},
  GridLines → None, GridLinesStyle → {}, ImageMargins → 0., ImagePadding → All,
  ImageSize → Automatic, ImageSizeRaw → Automatic, LabelStyle → {},
  MaxRecursion → Automatic, Mesh → Automatic, MeshFunctions → Automatic, MeshShading → None,
  MeshStyle → Automatic, Method → Automatic, PerformanceGoal :→ $PerformanceGoal,
  PlotLabel → None, PlotLegends → None, PlotPoints → Automatic, PlotRange → Automatic,
  PlotRangeClipping → True, PlotRangePadding → Automatic, PlotRegion → Automatic,
  PlotStyle → Automatic, PlotTheme :→ $PlotTheme, PreserveImageOptions → Automatic,
  Prolog → {}, RegionFunction → (True &), RotateLabel → True, TargetUnits → Automatic,
  TextureCoordinateFunction → Automatic, TextureCoordinateScaling → Automatic,
  Ticks → Automatic, TicksStyle → {}, WorkingPrecision → MachinePrecision}
```

```
Set[a, b]
```

```
b
```

```
a
```

```
b
```

```
Set[Evaluate[a], c]
```

```
c
```

```
a
```

```
c
```

```
b
```

```
c
```

```
?? Set
```

lhs = *rhs* evaluates *rhs* and assigns the result to be the value of *lhs*. From then on, *lhs* is replaced by *rhs* whenever it appears.

$\{l_1, l_2, \ldots\} = \{r_1, r_2, \ldots\}$ evaluates the r_i, and assigns the results to be the values of the corresponding l_i. ≫

```
Attributes[Set] = {HoldFirst, Protected, SequenceHold}
```

```
With[{A = {{-2, 0}, {0, -1}}},
 Evaluate[Table[MatrixExp[t A].{Cos[θ], Sin[θ]}, {θ, π/4, 2π, π/4}]]]
```

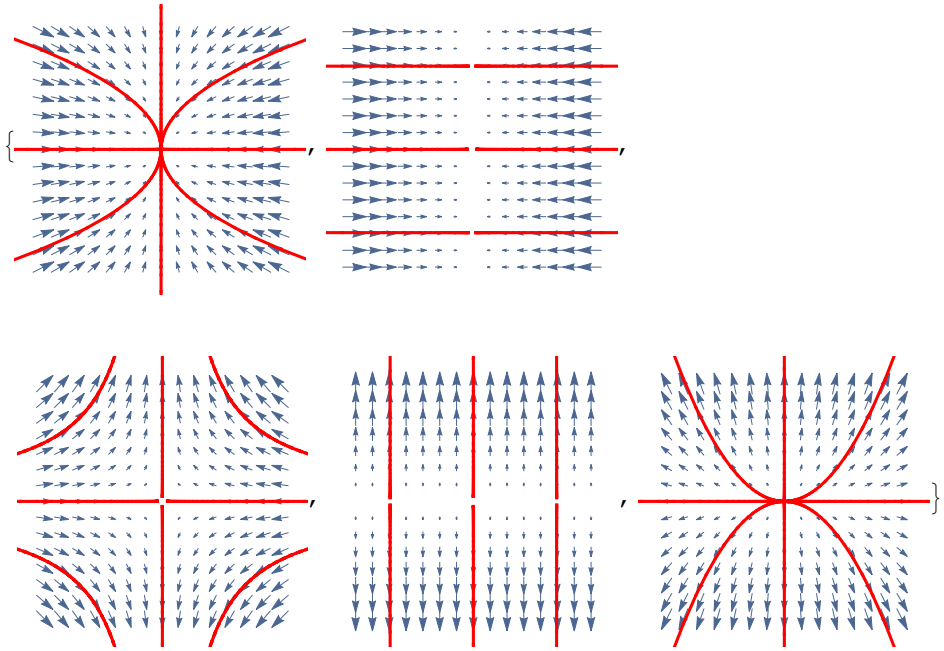
$$\left\{\left\{\left\{\frac{e^{-2t}}{\sqrt{2}}\right\}, \left\{\frac{e^{-t}}{\sqrt{2}}\right\}\right\}, \{\{0\}, \{e^{-t}\}\}, \left\{\left\{-\frac{e^{-2t}}{\sqrt{2}}\right\}, \left\{\frac{e^{-t}}{\sqrt{2}}\right\}\right\}, \{\{-e^{-2t}\}, \{0\}\},\right.$$
$$\left.\left\{\left\{-\frac{e^{-2t}}{\sqrt{2}}\right\}, \left\{-\frac{e^{-t}}{\sqrt{2}}\right\}\right\}, \{\{0\}, \{-e^{-t}\}\}, \left\{\left\{\frac{e^{-2t}}{\sqrt{2}}\right\}, \left\{-\frac{e^{-t}}{\sqrt{2}}\right\}\right\}, \{\{e^{-2t}\}, \{0\}\}\right\}$$

```
A
```

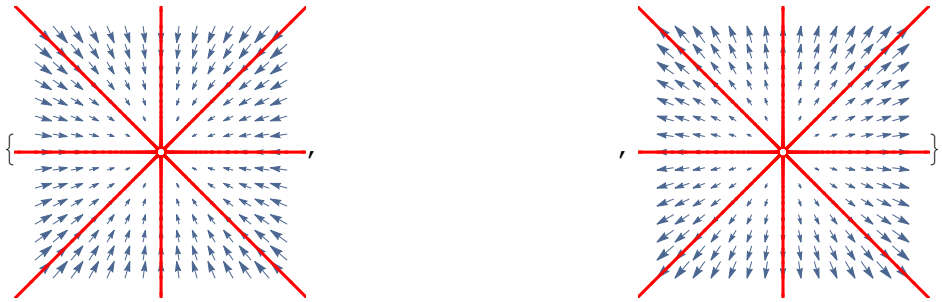
A

```
PP[A_] := Show[VectorPlot[A.{x, y}, {x, -1, 1}, {y, -1, 1}, Frame → None],
  ParametricPlot[Evaluate[Table[MatrixExp[t A].{Cos[θ], Sin[θ]}, {θ, π/4, 2π, π/4}]],
   {t, -π, π}, ColorFunction → (Red &)],
  ImageSize → 150]
```

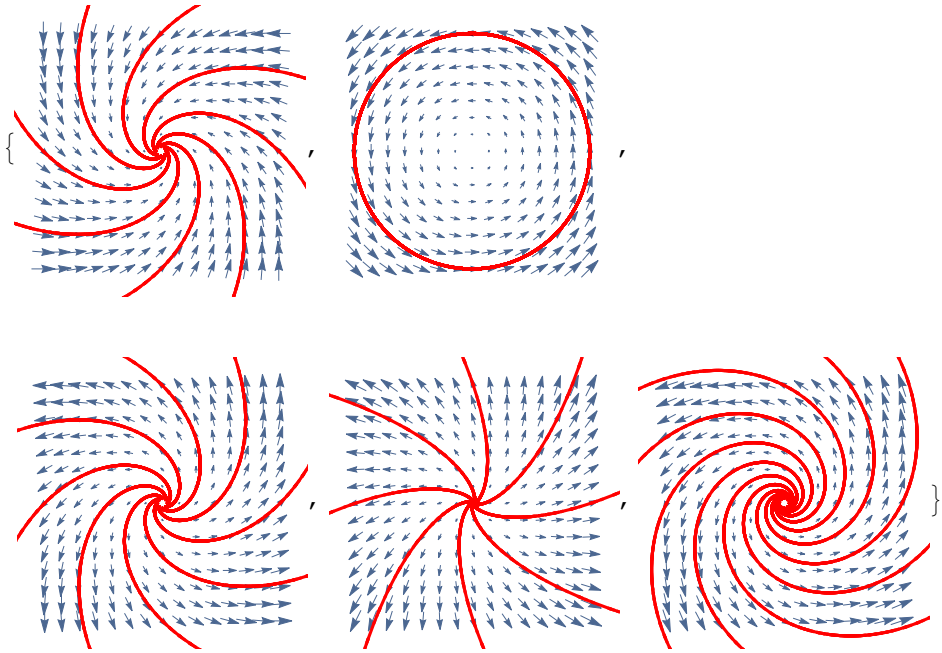
```
PP /@ {{{-2, 0}, {0, -1}}, {{-1, 0}, {0, 0}}, {{-1, 0}, {0, 1}}, {{0, 0}, {1, 0}}, {{1, 0}, {0, 2}}}
```



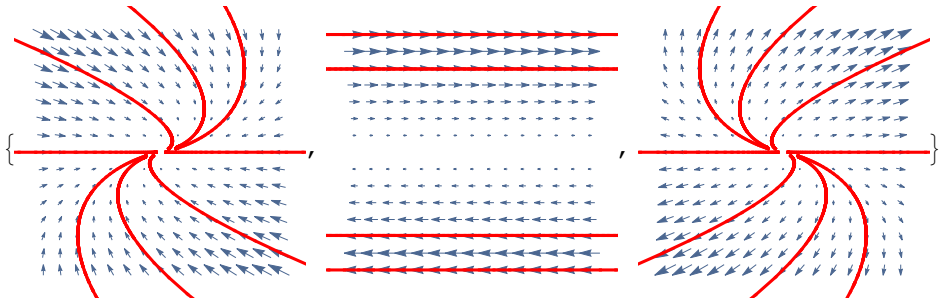
```
PP /@ {( -1  0 ), ( 0  0 ), ( 1  0 )}
      (  0 -1 )   ( 0  0 )   ( 0  1 )
```



```
PP /@ {( -1 -1 ), ( 0 -1 ), ( 1 -1 ), ( 3 -1 ), ( 1 -2 )}
      (  1 -1 )   ( 1  0 )   ( 1  1 )   ( 1  3 )   ( 2  1 )
```



```
PP /@ {( -1  1 ), ( 0  1 ), ( 1  1 )}
      (  0 -1 )   ( 0  0 )   ( 0  1 )
```



Quadratic Phase Portraits

```
RandomQuadratic[] :=
  (a x^2 + Sqrt[2] b x y + c y^2) /. a | b | c :> RandomVariate[NormalDistribution[]];
RandomQuadratic[n_] := Table[RandomQuadratic[], {n}]
```

```
RandomQuadratic[2]
```

$\{2.08457\, x^2 + 0.255033\, x\, y - 0.646763\, y^2,\ 1.88009\, x^2 - 3.74949\, x\, y + 1.19981\, y^2\}$

```
Dynamic[$WorkingOn]
```

```
$WorkingOn
```

```
Off[NDSolve::ndsz, InterpolatingFunction::dmval];
quads = {};
Rasterize[GraphicsGrid[Table[
   quad = RandomQuadratic[2];
   AppendTo[quads, $WorkingOn = {{i, j} -> quad}];
   Show[Join[
     {VectorPlot[quad/Norm[quad]^(3/4), {x, -1, 1}, {y, -1, 1}, Frame -> None]},
     Table[
      eqns = Join[
        Thread[{x'[t], y'[t]} == (quad /. {x -> x[t], y -> y[t]})],
        {x[0] == RandomReal[{-1, 1}], y[0] == RandomReal[{-1, 1}]}
       ];
      sol = NDSolve[eqns, {x, y}, {t, -1, 1}];
      ParametricPlot[
       Evaluate[{x[t], y[t]} /. sol],
       {t, -1, 1},
       PlotRange -> {{-1, 1}, {-1, 1}}, ColorFunction -> (Red &)
      ],
      {100}
     ]
    ]],
   {i, 5}, {j, 8}
  ]], ImageSize -> 960]
```

NDSolve::ndsz : At t == −0.564195, step size is effectively zero; singularity or stiff system suspected. >>

InterpolatingFunction::dmval :

Input value {−1.} lies outside the range of data in the interpolating function. Extrapolation will be used. >>

InterpolatingFunction::dmval :

Input value {−0.999959} lies outside the range of data in the interpolating function. Extrapolation will be used. >>

InterpolatingFunction::dmval :

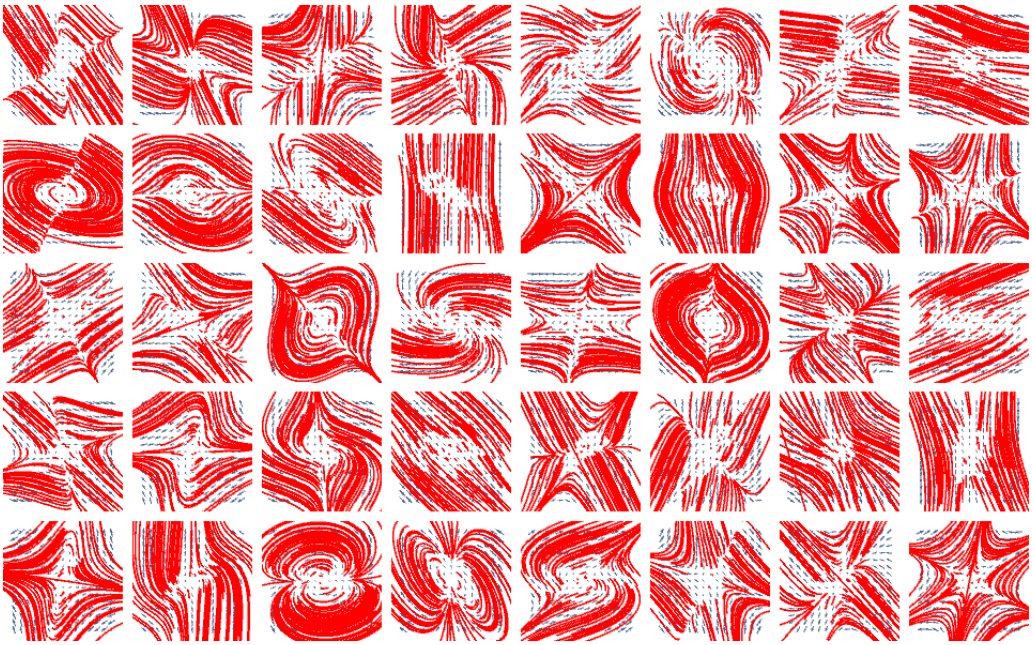
Input value {−0.999959} lies outside the range of data in the interpolating function. Extrapolation will be used. ≫

General::stop : Further output of InterpolatingFunction::dmval will be suppressed during this calculation. ≫

NDSolve::ndsz : At t == −0.622598, step size is effectively zero; singularity or stiff system suspected. ≫

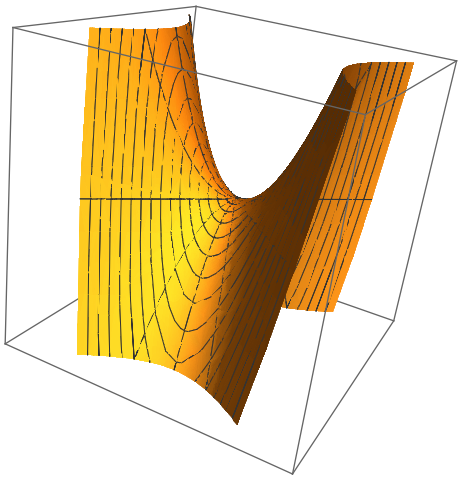
NDSolve::ndsz : At t == 0.47818703543930857`, step size is effectively zero; singularity or stiff system suspected. ≫

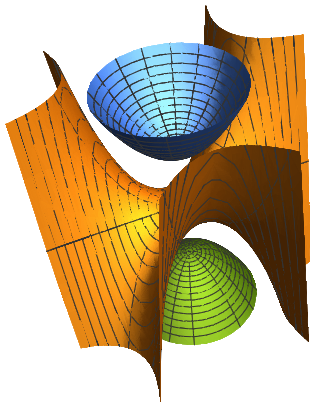
General::stop : Further output of NDSolve::ndsz will be suppressed during this calculation. ≫



The Saddle and the Monkey Saddle

```
GraphicsRow @@ {
 ParametricPlot3D[
  {r Cos[t], r Sin[t], 3 r^2 Cos[2 t]},
  {r, 0, 1}, {t, 0, 2 Pi},
  PlotRange → {-1, 1}, Axes → False,
  PlotPoints → {10, 30},
  ViewPoint → 0.8 {2.4, -1.2, 1.5}
 ],
 ParametricPlot3D[{
   {r Cos[t], r Sin[t], 3 r^3 Cos[3 t]},
   {r Cos[t], r Sin[t], 1 + r^2} / 2,
   {r Cos[t], r Sin[t], -1 - r^2} / 2
  }, {r, 0, 1}, {t, 0, 2 Pi},
  PlotRange → {-1, 1}, Boxed → False,
  Axes → False,
  PlotPoints → {10, 30},
  ViewPoint → 0.8 {2.4, -1.2, 1.5}]
}
```





```
z = Assuming[x > 0 ∧ y > 0, Re[(x + i y)^3 // Expand]]
```

$$-3\,\text{Im}\left[x^2\,y\right]+\text{Im}\left[y^3\right]+\text{Re}\left[x^3-3\,x\,y^2\right]$$

```
z = x^3 - 3 x y^2
```

$$x^3-3\,x\,y^2$$

```
quad = {-∂_y z, ∂_x z}
```

$$\left\{6\,x\,y,\ 3\,x^2-3\,y^2\right\}$$

```
Show[Join[
  {VectorPlot[quad/Norm[quad]^(3/4), {x, -1, 1}, {y, -1, 1}, Frame → None]},
  Table[
   eqns = Join[
     Thread[{x'[t], y'[t]} == (quad /. {x → x[t], y → y[t]})],
     {x[0] == RandomReal[{-1, 1}], y[0] == RandomReal[{-1, 1}]}
    ];
   sol = NDSolve[eqns, {x, y}, {t, -1, 1}];
   ParametricPlot[
    Evaluate[{x[t], y[t]} /. sol],
    {t, -1, 1},
    PlotRange → {{-1, 1}, {-1, 1}}, ColorFunction → (Red &)
   ],
   {100}
  ]
 ]]
```

NDSolve::ndsz : At t == −0.422564, step size is effectively zero; singularity or stiff system suspected. ≫

NDSolve::ndsz : At t == 0.3276507605938512`, step size is effectively zero; singularity or stiff system suspected. ≫

InterpolatingFunction::dmval :

Input value {−1.} lies outside the range of data in the interpolating function. Extrapolation will be used. ≫

InterpolatingFunction::dmval :

Input value {−0.999959} lies outside the range of data in the interpolating function. Extrapolation will be used. ≫

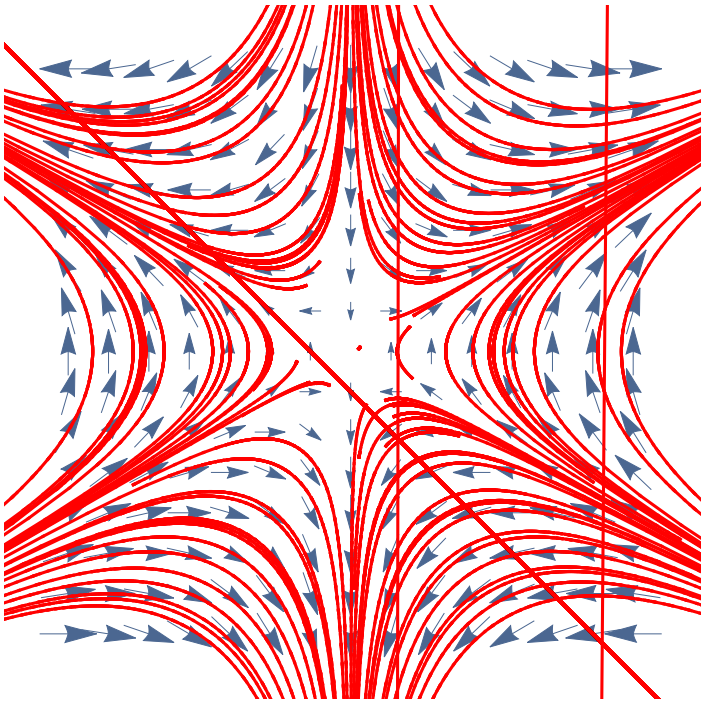
InterpolatingFunction::dmval :

Input value {−0.999959} lies outside the range of data in the interpolating function. Extrapolation will be used. ≫

General::stop : Further output of InterpolatingFunction::dmval will be suppressed during this calculation. ≫

NDSolve::ndsz : At t == −0.398864, step size is effectively zero; singularity or stiff system suspected. ≫

General::stop : Further output of NDSolve::ndsz will be suppressed during this calculation. ≫



The Willerton Fish

Willerton: It is amuzing to plot the values of v_2 against the values of v_3 on a large sample of knots.

```
<< KnotTheory`

GD[L_] :=
  GD @@ PD[L] /. X[i_, j_, k_, l_] :> If[PositiveQ[X[i, j, k, l]], Ap[l, i], Am[j, i]];
Column[GD /@ AllKnots[{3, 6}]]
```

From Polyak-Viro's "Guass Diagram Formulas for Vassiliev Invariants", IMRN 11 (1994) 445-453:

3.B THEOREM 1. *If G is any based Gauss diagram of a knot K then*

$$v_2(K) = \left\langle \text{  }, G \right\rangle, \tag{4}$$

where $v_2(K)$ is the Vassiliev invariant of degree 2 which takes values 0 on the unknot and 1 on a trefoil.

4.A THEOREM 2. *If G is a Gauss diagram of a knot K then*

$$v_3(K) = \left\langle \frac{1}{2}\left[\text{  }\right] + \left[\text{  }\right], G \right\rangle, \tag{5}$$

where $v_3(K)$ is the Vassiliev invariant of degree 3 which takes values 0 on the unknot, $+1$ on the right trefoil and -1 on the left trefoil.