Pensieve header: Implementing KH, day 3.

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Import["http://www.math.toronto.edu/~drorbn/papers/Categorification/QRG.pdf"]
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AQuickReferenceGuidetoKhovanov'sCategori cationoftheJonesPolynomial

DrorBar-Natan,June12,2002

TheKaufimanBracket: $\langle\emptyset\rangle=1$; $\langle \bigcirc L\rangle=(q+q^{-1})\langle L\rangle$; $\langle \times \rangle = \langle \asymp_{0-\text{smoothing}}\rangle - q\langle)(_{1-\text{smoothing}}\rangle$.

TheJonesPolynomial: $J(L)=(-1)^{n_-}q^{n_+-2n_-}\langle L\rangle$,where(n_+,n_-)count(,)crossings.

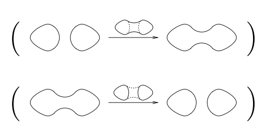
Khovanov'sconstruction: $[\![L]\!]$ achaincomplexofgraded $\mathbb{Z}$-modules;

$$[\![\emptyset]\!]=0\to\underset{\text{height}0}{\mathbb{Z}}\to 0;\qquad [\![\bigcirc L]\!]=V\otimes[\![L]\!];\qquad [\![\times]\!]=\text{Flatten}\left(0\to\underset{\text{height}0}{[\![\asymp]\!]}\to\underset{\text{height}1}{[\![)(]\!]\{1\}}\to 0\right);$$

$$\mathcal{H}(L)=\mathcal{H}(\mathcal{C}(L)=[\![L]\!][-n_-]\{n_+-2n_-\})$$

$$V=\text{span}\langle v_+,v_-\rangle;\ \deg v_\pm=\pm1;\quad q\dim V=q+q^{-1}\quad\text{with}\quad q\dim\mathcal{O}:=\sum_m q^m\dim\mathcal{O}_m;$$

$$\mathcal{O}\{l\}_m:=\mathcal{O}_{m-l}\quad\text{so}\quad q\dim\mathcal{O}\{l\}=q^l\,q\dim\mathcal{O};\qquad\cdot[s]:\text{heightshiftby } s;$$

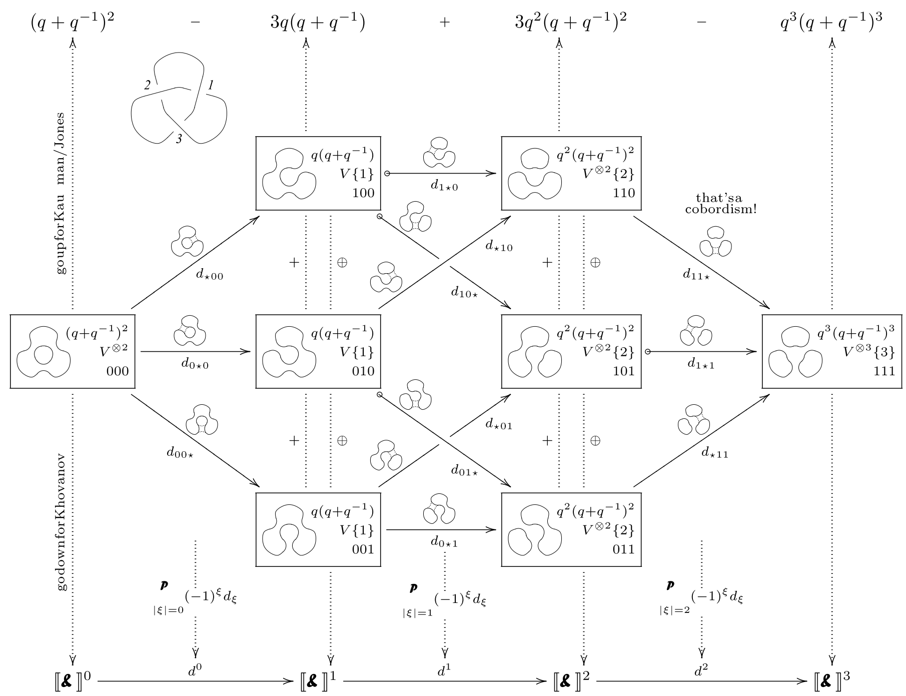


$\longrightarrow(V\otimes V\xrightarrow{m}V)$ $\quad m:\begin{cases}v_+\otimes v_-\mapsto v_- & v_+\otimes v_+\mapsto v_+\\ v_-\otimes v_+\mapsto v_- & v_-\otimes v_-\mapsto 0\end{cases}$

$\longrightarrow(V\to V\otimes V)$ $\quad :\begin{cases}v_+\mapsto v_+\otimes v_-+v_-\otimes v_+\\ v_-\mapsto v_-\otimes v_-\end{cases}$

That'saFrobenius Algebra!Anda (1+1)-dimensional TQFT!

Example: $\quad q^{-2}+1+q^2-q^6 \xrightarrow[\text{(with}(n_+,n_-)=(3,0))]{\cdot(-1)^{n_-}q^{n_+-2n_-}} q+q^3+q^5-q^9.$



(here$(-1)^\xi:=(-1)^{\sum_{i<j}\xi_i}$ if $\xi_j=\star$) $=$ $\quad [\![\&]\!]\xrightarrow[\text{(with}(n_+,n_-)=(3,0))]{\cdot[-n_-]\{n_+-2n_-\}}\mathcal{C}(\&)$.

Theorem1. ThegradedEulercharacteristicof $\mathcal{C}(L)$is $J(L)$.

Theorem2. Thehomology $\mathcal{H}(L)$isalinkinvariantandthussois $Kh_{\mathbb{F}}(L):=\sum_r t^r\,q\dim\mathcal{H}^r_{\mathbb{F}}(\mathcal{C}(L))$overany eld $\mathbb{F}$.

Theorem3. $\mathcal{H}(\mathcal{C}(L))$isstrictlystrongerthan $J(L)$: $\mathcal{H}(\mathcal{C}(5_1))\neq\mathcal{H}(\mathcal{C}(10_{132}))$whereas $J(5_1)=J(10_{132})$.

Conjecture1. $Kh_{\mathbb{Q}}(L)=q^{s-1}\left(1+q^2+(1+tq^4)Kh'\right)$ and $Kh_{\mathbb{F}_2}(L)=q^{s-1}(1+q^2)\left(1+(1+tq^2)Kh'\right)$ foreven $s=s(L)$andnon-negative-coe cientslaurentpolynomial $Kh'=Kh'(L)$.

Conjecture2. Foralternatingknots s isthesignatureand Kh' dependsonlyon tq^2.

References. Khovanov'sarXiv:math.QA/9908171andarXiv:math.QA/0103190andDBN's
http://www.ma.huji.ac.il/∼drorbn/papers/Categorification/.

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Road map:

1. Produce all the info in a “cube of smoothings”.

2. Produce a commuting cube of vector spaces and maps between them.
3. Produce a complex.
4. Compute homology.

```
K[3, 1] = Xm[1, 4, 2, 5] Xm[3, 6, 4, 1] Xm[5, 2, 6, 3];
K[5, 1] = Xm[1, 6, 2, 7] Xm[3, 8, 4, 9] Xm[5, 10, 6, 1] Xm[7, 2, 8, 3] Xm[9, 4, 10, 5];
K[10, 132] =
  Xm[5, 12, 6, 13] Xm[9, 16, 10, 17] Xm[11, 6, 12, 7] Xm[13, 20, 14, 1] Xm[15, 18, 16, 19]
   Xm[17, 10, 18, 11] Xm[19, 14, 20, 15] Xp[2, 8, 3, 7] Xp[4, 2, 5, 1] Xp[8, 4, 9, 3];
```

```
SetAttributes[p, Orderless];
SetAttributes[dx, Orderless];
α = 0;
ϵ /: ϵ^n_ /; n > 1 = 0;
t1 = Expand[
      dx[] K[10, 132] /. {
          (Xp | Xm)[i_, j_, k_, l_] :→ (++α;
            p[i, j] p[k, l] + dx[α] p[i, l] p[j, k] + ϵ e[α] X[i, j, k, l])
         } /. p[i_, j_] :→ p[i, j][Min[i, j]]
     ] //. p[i_, j_][m_] p[j_, k_][n_] :→ p[i, k][Min[m, n]] /. {
     X[i_, j_, k_, l_] p[i_, j_][m_] p[k_, l_][n_] :→ (c[m] c[n] → c[Min[m, n]]),
     X[i_, j_, k_, l_] p[i_, l_][m_] p[j_, k_][n_] :→ (c[Min[m, n]] → c[m] c[n])
    } /.
   p[i_, j_][m_]^-· :→ c[m] //. dx[i___] dx[j___] :→ dx[i, j]
```

```
c[1] c[2] c[6] c[10] dx[] + c[1] c[2] c[10] dx[1] + ⋯6141⋯ +
 ϵ c[1] c[3] c[6] c[10] dx[1, 2, 3, 4, 5, 6, 8, 9, 10] e[7] (c[14] → c[14] c[15])
```

large output | show less | show more | show all | set size limit...

```
SetAttributes[{p, dx}, Orderless];
S[K_] := Module[{α = 0, ϵ, X, Z},
  ϵ /: ϵ^- = 0;
  Z = Expand[
       dx[] K /. {
           (Xp | Xm)[i_, j_, k_, l_] :→ (++α;
             p[i, j] p[k, l] + dx[α] p[i, l] p[j, k] + ϵ e[α] X[i, j, k, l])
          } /. p[i_, j_] :→ p[i, j][Min[i, j]]
      ] //. p[i_, j_][m_] p[j_, k_][n_] :→ p[i, k][Min[m, n]] /. {
      X[i_, j_, k_, l_] p[i_, j_][m_] p[k_, l_][n_] :→ (c[m] c[n] → c[Min[m, n]]),
      X[i_, j_, k_, l_] p[i_, l_][m_] p[j_, k_][n_] :→ (c[Min[m, n]] → c[m] c[n])
     } /.
    p[i_, j_][m_]^-· :→ c[m] //. dx[i___] dx[j___] :→ dx[i, j];
  CoefficientList[Z, ϵ, {2}]
 ]
```

```
CoefficientList[(1 + x)^7, x, {4}]
```

```
{1, 7, 21, 35}
```

```
S[K[3, 1]] // Column
```

```
c[1] c[2] c[3] dx[] + c[1] c[3] dx[1] + c[1] c[2] dx[2] + c[1] c[2] dx[3] +
 c[1] dx[1, 2] + c[1] dx[1, 3] + c[1] dx[2, 3] + c[1] c[2] dx[1, 2, 3]
dx[2, 3] e[1] (c[1] → c[1] c[2]) + dx[1, 3] e[2] (c[1] → c[1] c[2]) +
 dx[1, 2] e[3] (c[1] → c[1] c[2]) + c[3] dx[] e[1] (c[1] c[2] → c[1]) +
 dx[2] e[1] (c[1] c[2] → c[1]) + dx[3] e[1] (c[1] c[2] → c[1]) +
 dx[3] e[2] (c[1] c[2] → c[1]) + dx[2] e[3] (c[1] c[2] → c[1]) +
 c[2] dx[] e[2] (c[1] c[3] → c[1]) + dx[1] e[2] (c[1] c[3] → c[1]) +
 dx[1] e[3] (c[1] c[3] → c[1]) + c[1] dx[] e[3] (c[2] c[3] → c[2])
```

```
({S0, S1} = S[K[10, 132]]) // Timing
```

```
{9.625, {c[1] c[2] c[6] c[10] dx[] + c[1] c[2] c[10] dx[1] + ⋯ 1021 ⋯ +
   c[1] c[3] c[6] c[10] c[14] c[15] dx[1, 2, 3, 4, 5, 6, 7, 8, 9, 10], ⋯ 1 ⋯ }}
```

large output | show less | show more | show all | set size limit...

```
({S0, S1} = S[K[3, 1]]) // Timing
```

```
{0.015625, {c[1] c[2] c[3] dx[] + c[1] c[3] dx[1] + c[1] c[2] dx[2] +
   c[1] c[2] dx[3] + c[1] dx[1, 2] + c[1] dx[1, 3] + c[1] dx[2, 3] + c[1] c[2] dx[1, 2, 3],
  dx[2, 3] e[1] (c[1] → c[1] c[2]) + dx[1, 3] e[2] (c[1] → c[1] c[2]) +
   dx[1, 2] e[3] (c[1] → c[1] c[2]) + c[3] dx[] e[1] (c[1] c[2] → c[1]) +
   dx[2] e[1] (c[1] c[2] → c[1]) + dx[3] e[1] (c[1] c[2] → c[1]) +
   dx[3] e[2] (c[1] c[2] → c[1]) + dx[2] e[3] (c[1] c[2] → c[1]) +
   c[2] dx[] e[2] (c[1] c[3] → c[1]) + dx[1] e[2] (c[1] c[3] → c[1]) +
   dx[1] e[3] (c[1] c[3] → c[1]) + c[1] dx[] e[3] (c[2] c[3] → c[2])}}
```

```
V0 = List @@ Expand[S0 /. c[i_] :> vp[i] + vm[i]]
```

```
{dx[1, 2] vm[1], dx[1, 3] vm[1], dx[2, 3] vm[1], dx[2] vm[1] vm[2], dx[3] vm[1] vm[2],
 dx[1, 2, 3] vm[1] vm[2], dx[1] vm[1] vm[3], dx[] vm[1] vm[2] vm[3], dx[1, 2] vp[1],
 dx[1, 3] vp[1], dx[2, 3] vp[1], dx[2] vm[2] vp[1], dx[3] vm[2] vp[1], dx[1, 2, 3] vm[2] vp[1],
 dx[1] vm[3] vp[1], dx[] vm[2] vm[3] vp[1], dx[2] vm[1] vp[2], dx[3] vm[1] vp[2],
 dx[1, 2, 3] vm[1] vp[2], dx[] vm[1] vm[3] vp[2], dx[2] vp[1] vp[2], dx[3] vp[1] vp[2],
 dx[1, 2, 3] vp[1] vp[2], dx[] vm[3] vp[1] vp[2], dx[1] vm[1] vp[3], dx[] vm[1] vm[2] vp[3],
 dx[1] vp[1] vp[3], dx[] vm[2] vp[1] vp[3], dx[] vm[1] vp[2] vp[3], dx[] vp[1] vp[2] vp[3]}
```

```
Length@V0
```

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30
```

```
dx[3] e[2] (c[1] c[2] → c[1])
```

```
Clear[d]
```

```
d[dx[αs___] e[β_] cs_. (c[i_] c[j_] → c[k_])] := {
  dx[αs] vp[i] vp[j] → dx[αs, β] vp[k],
  dx[αs] vm[i] vp[j] → dx[αs, β] vm[k],
  dx[αs] vp[i] vm[j] → dx[αs, β] vm[k],
  dx[αs] vm[i] vm[j] → 0,
  dx[βs___] /; {βs} =!= {αs} :→ 0
 }
```

```
V0 /. d[dx[3] e[2] (c[1] c[2] → c[1])]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, dx[2, 3] vm[1], 0, 0,
 0, 0, dx[2, 3] vm[1], 0, 0, 0, dx[2, 3] vp[1], 0, 0, 0, 0, 0, 0, 0, 0}
```