Sample boundary target measure space; failure to construct sample interior source measure.

In this notebook we (attempt to) construct the desired target and source sample spaces for our semicoupling problem. Our spaces are derived from the so-called rational Borel-Serre bordification of the symmetric space SO(2,R) \ SL(2,R) relative to the right-action of SL(2,Z). The target space Y is obtained as the topological boundary resulting from the SL(2,Z)-equivariant excision of the Poincare disk by rational horospheres.

We are interested in the uniform haar measure supported on the horospheres, i.e. boundary components of Y. These horospheres are one-dimensional unipotent orbits, and their uniform measure can be well approximated by discrete measures. On the contrary, to approximate the uniform area measure on the interior source space, we met a difficulty. In the euclidean case, if one has a small interior epsilon ball $B(\epsilon)$, then say using small hexagonal grids, one obtains discrete measures which converge to the uniform lebesgue measure restricted to the ball B(ϵ). However, in the hyperbolic, or CAT(0) case, the uniform measure on balls B(ϵ) is concentrated on the boundary of the ball, and the hexagonal grid is no longer a sound or meaningful approximation. We do not know how to discretize balls B(ϵ) in the hyperbolic disk! Thus we have been unable to locally uniformly sample our source space. This failure is rather shameful.

One attempt at such a uniform sample was: it is understood that an SL(2,Z)-orbit becomes asymptotically equidistributed for large radius balls, i.e. the number of points in a given SL(2,Z)-orbit contained in a ball B(R) of large radius is equivalent to the uniform area of the ball. A naive idea was to rescale a given local SL(2,Z) orbit to lie within, say, a unit ball B(1), and prove that the correspondant renormalized atomic measure converges to the uniform area measure on B(1). We attempted this rescaling by taking successive midpoints of the SL(2,Z)-orbit. Our midpoint function, is however, corrupted.

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u[1] = {{1, 1}, {0, 1}};
u[2] = {{1, -1}, {0, 1}};
u[3] = {{1, 0}, {1, 1}};
u[4] = {{1, 0}, {-1, 1}};
u[5] = {{1, 0}, {0, 1}};
p[0] = Table[u[j], {j, 1, 5}];
p[1] = Tuples[Table[u[j], {j, 1, 5}], 2] /. {x_List, y_List} → x.y // Union;
p[2] = Tuples[{p[1], p[0]}] /. {x_List, y_List} → x.y // Union;
p[j_Integer] := p[j] = Tuples[{p[j - 1], p[0]}] /. {x_List, y_List} :→ x.y // Union;
pp[n_Integer] := GatherBy[p[n], Dot[#, {1, 0}] &];
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```
hp[m_List] :=
  hp[m] = N@{(m[[1, 1]] m[[1, 2]] + m[[2, 1]] m[[2, 2]]) / (m[[1, 1]]^2 + m[[2, 1]]^2),
     1 / (m[[1, 1]]^2 + m[[2, 1]]^2)};
cay[{a_Real, b_Real}] := cay[{a, b}] = {(a^2 + b^2 - 1) / (a^2 + (b + 1)^2),
    (-2 a) / (a^2 + (b + 1)^2) };
comp[m_List] := comp[m] = cay[#] &@hp[m];

k[k1_Real] := RotationMatrix[k1];
a[a1_Real] := a[a1] = {{Exp[-a1], 0}, {0, Exp[a1]}};

horo[a1_Real, n1_Integer: 30] :=
  horo[a1, n1] = Range[-n1, n1] /. n_Integer -> a[a1].MatrixPower[u[1], n];

adj[k_List, x_List] := adj[k, x] = k.x.Inverse[k]


scale[v_List, a_Real] := v.{{a^(-1), 0}, {0, a^(-1)}};
id = {{1, 0}, {0, 1}};
ref = {{1, 0}, {0, -1}};
j = {{0, -1}, {1, 0}};

nml[m_List] := nml[m] = Transpose[{m.{1, 0}, j.m.{1, 0}}];

normal[m_List] := normal[m] =
   {{N[Norm[nml[m][[1]], 2]^(-1)], 0}, {0, N[Norm[nml[m][[1]], 2]^(-1)]}}.nml[m];

iwa[m_List] := iwa[m] = normal[m].
    {{(Inverse[normal[m]].m)[[1, 1]], 0}, {0, (Inverse[normal[m]].m)[[2, 2]]}};


approxcomp[xsym_List, n_Integer: 10] := approxcomp[xsym, n] = {
     scale[N@MatrixPower[xsym, n, {1, 0}],
      Norm[N@MatrixPower[N@xsym, n, {1, 0}], 2]],
     scale[N@MatrixPower[xsym, -n, {1, 0}], Norm[
       N@MatrixPower[N@xsym, -n, {1, 0}], 2]]} /. x_List /; Re[Det[x]] < 0 :→ x.ref;
adj[k_List, x_List] := adj[k, x] = k.x.Inverse[k] ;
symm[x_List] := Transpose[x].x;
k[ y_List, x_List, n_Integer: 10] :=
  k[y, x, n] = approxcomp[symm[N@y.N@Inverse[x]], n];
torus[y_List, x_List, n_Integer: 10] :=
  torus[y, x, n] = adj[k[y, x, n], symm[N@y.N@Inverse[x]]];
```

```
Module[{k},
  k = 1;
  Do[g[k_Integer] := N@pp[5][[k, 1]]; k = k + 1, {Length[pp[5]]}]];
```

Sample boundary of equivariant excision by rational horospheres:

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ListPlot[Table[comp[#] & /@ (Dot[#, g[k]] & /@ horo[1.0, 15]), {k, 1, 190}],
 AspectRatio → Automatic]
```