Knots

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In this program, I experimented with Mathematica’s built-in functions to present beautiful pictures of various knots and tried to produce some myself.

KnotData

KnotData can be used to get all knots of crossing numbers up to 10 crossings as follows:

```
KnotData[All]
```

Basic Knots

KnotData is also used to present a picture of some of the knots. For example:

```
KnotData["Unknot"] // Timing
```

```
KnotData["Trefoil"] // Timing
```

```
KnotData["FigureEight"] // Timing
```

```
KnotData["SolomonSeal"] // Timing
```

```
KnotData["Stevedore"] // Timing
```

GraphicsGrid

GraphicsGrid is a Mathematica code that can put pictures in a grid form. However, the code does not work with 3D. So I experimented and tried to add “3D” to the code and it worked.

```
GraphicsGrid3D[KnotData["Unknot"], KnotData["Trefoil"], KnotData["FigureEight"],
  KnotData["SolomonSeal"], KnotData["Stevedore"]] // Timing
```

Torus Knots

Mathematica has a build-in code for the torus knots. You only need to decide on the values of p and q in the (p,q)-torus knot. The following example is the (2,3)-torus knot which is the trefoil knot.

```
KnotData[{"TorusKnot", {2, 3}}] // Timing
```

Torus Knots up to 10 crossings

KnotData [“Torus”] is used to show the torus knots up to 10 crossings.

```
KnotData["Torus"]
GraphicsGrid3D[KnotData[{"TorusKnot", {1, 1}}],
  KnotData[{"TorusKnot", {2, 3}}], KnotData[{"TorusKnot", {2, 5}}],
  KnotData[{"TorusKnot", {2, 7}}], KnotData[{"TorusKnot", {3, 4}}],
  KnotData[{"TorusKnot", {2, 9}}], KnotData[{"TorusKnot", {3, 5}}]] // Timing
```

(2,q) - Torus Knots

Here, we present pictures of torus knots with p=2 and q the coprime numbers to p up to 47.

```
GraphicsGrid3D[KnotData[{"TorusKnot", {2, 3}}],
  KnotData[{"TorusKnot", {2, 5}}], KnotData[{"TorusKnot", {2, 7}}],
  KnotData[{"TorusKnot", {2, 9}}], KnotData[{"TorusKnot", {2, 11}}],
  KnotData[{"TorusKnot", {2, 13}}], KnotData[{"TorusKnot", {2, 15}}],
  KnotData[{"TorusKnot", {2, 17}}], KnotData[{"TorusKnot", {2, 19}}],
  KnotData[{"TorusKnot", {2, 21}}], KnotData[{"TorusKnot", {2, 23}}],
  KnotData[{"TorusKnot", {2, 25}}], KnotData[{"TorusKnot", {2, 27}}],
  KnotData[{"TorusKnot", {2, 29}}], KnotData[{"TorusKnot", {2, 31}}],
  KnotData[{"TorusKnot", {2, 33}}], KnotData[{"TorusKnot", {2, 35}}],
  KnotData[{"TorusKnot", {2, 37}}], KnotData[{"TorusKnot", {2, 39}}],
  KnotData[{"TorusKnot", {2, 41}}], KnotData[{"TorusKnot", {2, 43}}],
  KnotData[{"TorusKnot", {2, 45}}], KnotData[{"TorusKnot", {2, 47}}]] // Timing
```

(3, q) - Torus Knots

Here, we present pictures of torus knots with p = 3 and q the coprime numbers to p up to 32.

```
GraphicsGrid3D[KnotData[{"TorusKnot", {3, 4}}], KnotData[{"TorusKnot", {3, 5}}],
  KnotData[{"TorusKnot", {3, 7}}], KnotData[{"TorusKnot", {3, 8}}],
  KnotData[{"TorusKnot", {3, 10}}], KnotData[{"TorusKnot", {3, 11}}],
  KnotData[{"TorusKnot", {3, 13}}], KnotData[{"TorusKnot", {3, 14}}],
  KnotData[{"TorusKnot", {3, 16}}], KnotData[{"TorusKnot", {3, 17}}],
  KnotData[{"TorusKnot", {3, 19}}], KnotData[{"TorusKnot", {3, 20}}],
  KnotData[{"TorusKnot", {3, 22}}], KnotData[{"TorusKnot", {3, 23}}],
  KnotData[{"TorusKnot", {3, 25}}], KnotData[{"TorusKnot", {3, 26}}],
  KnotData[{"TorusKnot", {3, 28}}], KnotData[{"TorusKnot", {3, 29}}],
  KnotData[{"TorusKnot", {3, 31}}], KnotData[{"TorusKnot", {3, 32}}]] // Timing
```

(4, q) - Torus Knots

Here, we present pictures of torus knots with p = 4 and q the coprime numbers to p up to 31.

```
GraphicsGrid3D[KnotData[{"TorusKnot", {4, 5}}], KnotData[{"TorusKnot", {4, 7}}],
  KnotData[{"TorusKnot", {4, 9}}], KnotData[{"TorusKnot", {4, 11}}],
  KnotData[{"TorusKnot", {4, 13}}], KnotData[{"TorusKnot", {4, 15}}],
  KnotData[{"TorusKnot", {4, 17}}], KnotData[{"TorusKnot", {4, 19}}],
  KnotData[{"TorusKnot", {4, 21}}], KnotData[{"TorusKnot", {4, 23}}],
  KnotData[{"TorusKnot", {4, 25}}], KnotData[{"TorusKnot", {4, 27}}],
  KnotData[{"TorusKnot", {4, 29}}], KnotData[{"TorusKnot", {4, 31}}]] // Timing
```

(5, q) - Torus Knots

Here, we present pictures of torus knots with p = 5 and q the coprime numbers to p up to 34.

```
GraphicsGrid3D[KnotData[{"TorusKnot", {5, 6}}], KnotData[{"TorusKnot", {5, 7}}],
  KnotData[{"TorusKnot", {5, 8}}], KnotData[{"TorusKnot", {5, 9}}],
  KnotData[{"TorusKnot", {5, 11}}], KnotData[{"TorusKnot", {5, 12}}],
  KnotData[{"TorusKnot", {5, 13}}], KnotData[{"TorusKnot", {5, 14}}],
  KnotData[{"TorusKnot", {5, 16}}], KnotData[{"TorusKnot", {5, 17}}],
  KnotData[{"TorusKnot", {5, 18}}], KnotData[{"TorusKnot", {5, 19}}],
  KnotData[{"TorusKnot", {5, 21}}], KnotData[{"TorusKnot", {5, 22}}],
  KnotData[{"TorusKnot", {5, 23}}], KnotData[{"TorusKnot", {5, 24}}],
  KnotData[{"TorusKnot", {5, 26}}], KnotData[{"TorusKnot", {5, 27}}],
  KnotData[{"TorusKnot", {5, 28}}], KnotData[{"TorusKnot", {5, 29}}],
  KnotData[{"TorusKnot", {5, 31}}], KnotData[{"TorusKnot", {5, 32}}],
  KnotData[{"TorusKnot", {5, 33}}], KnotData[{"TorusKnot", {5, 34}}]] // Timing
```

Torus Knots using my own Codes

Here, I tried to present pictures of torus knots using different methods. I also compared the speed of each of them with the built-in one.

```
r = Cos[q t] + 2;
x = r * Cos[p t];
y = r * Sin[p t];
z = -Sin[q t];
p = 2; q = 5;
Show[ParametricPlot3D[{x, y, z}, {t, 0, 2 π},
    ViewPoint → {0, 0, 100}, Boxed → False, Axes → False] /.
   Line[a_] :→ Tube[a, 0.2], PlotRange → All, ImageSize → 300] // Timing
```

```
torusKnot[p_, q_, t_] := With[{r = Cos[q t] + 2}, {r Cos[p t], r Sin[p t], Sin[q t]}]
pts = Table[torusKnot[2, 5, t], {t, 0, 2 π, π / 100}];
Graphics3D[{CapForm[None], Lighter@Red,
   Tube[pts, 0.2], FaceForm[None, Glow[White]], Tube[pts, 0.15]},
  ViewPoint → {0, 0, 100}, Boxed → False] // Timing
```

```
KnotData[{"TorusKnot", {2, 5}}] // Timing
```

Trefoil Knots

Here, I present pictures of the trefoil knots using different methods. I also compare the speed of each one with the buit-in.

```
torusKnot[p_, q_, t_] := With[{r = Cos[q t] + 2}, {r Cos[p t], r Sin[p t], Sin[q t]}]
pts = Table[torusKnot[2, 3, t], {t, 0, 2 π, 2 π / 200}];
Graphics3D[{CapForm[None], Lighter@Orange,
   Tube[pts, 0.2], FaceForm[None, Glow[White]], Tube[pts, 0.15]},
  ViewPoint → {0, 0, 100}, Boxed → False] // Timing
```

```
KnotData[{"TorusKnot", {2, 3}}] // Timing
```

```
trefoil[t_] = {Sin[3 t], Sin[t] + 2 Sin[2 t], Cos[t] - 2 Cos[2 t]};
Show[ParametricPlot3D[trefoil[t],
    {t, 0, 2 π}, ViewPoint → {25, 0, 0}, Boxed → False, Axes → False] /.
   Line[pts_] :→ Tube[pts, 0.2], PlotRange → All, ImageSize → 380] // Timing
KnotData["Trefoil"] // Timing
```

Trefoil Knots’ Planar Diagram

Mathematica has a buit-in code to present the knot in the planar diagram form.

```
KnotData["Trefoil", "KnotDiagram"] // Timing
```

Coding the Planar Diagram of the Trefoil Knot

Here, I present pictures of the planar diagram of the trefoil knot. I also compare the speed of each one with the buit-in.

```
trefoil2[t_] = {Sin[t] + 2 Sin[2 t], Cos[t] - 2 Cos[2 t]};
{diag, {s}} = Reap[ParametricPlot[{Sin[t] + 2 Sin[2 t], Cos[t] - 2 Cos[2 t]},
    {t, 0, 2 π}, Axes → False, EvaluationMonitor :→ Sow[t], ImageSize → 380]];
s = Sort[s];
diag // Timing
```

```
{xp[t_], yp[t_]} = trefoil2'[t];
n[t_] = {yp[t], -xp[t]};
n[t_] = Simplify[n[t] / Norm[n[t]], Element[t, Reals]];
gap = 0.15;
strip[{t1_, t2_}] :=
  {{White, EdgeForm[White], Polygon[{trefoil2[t1] + gap * n[t1], trefoil2[t2] +
       gap * n[t2], trefoil2[t2] - gap * n[t2], trefoil2[t1] - gap * n[t1]}]},
   {Thickness[0.005], Line[{trefoil2[t1], trefoil2[t2]}]}};
ps = Table[{trefoil[s[[i]]], {s[[i]], s[[i + 1]]}}, {i, 1, Length[s] - 1}];
xsort = Last /@ SortBy[ps, #[[1, 1]] &];
Graphics[strip /@ xsort, ImageSize → 380] // Timing
```

Trefoil Knots using Arrow with BSplineCurve

Here, I present pictures of the trefoil knots using a curved arrow. Also notice the timings and compare them to the previous ones.

```
Graphics3D[
 {Arrow[BSplineCurve[{{0, 0, 0}, {1, 1, 1}, {2, -1, 1}, {3, 0, 2}}]]}, Boxed → False]
Graphics3D[{Arrow[Tube[BSplineCurve[{{0, 0, 0}, {2, .3, 1}, {1, -.5, 2}, {-1, .5, 2},
      {-2, .3, 1}, {0, -1.5, 0}, {1, 1, 2}, {0, 0, 4}, {-1, -1, 2}, {0, 0, 0}}], .15]]},
  Boxed → False, ViewPoint → {0, -60, 30}, PlotRange → All,
  ImageSize → 380] // Timing
```

```
tref = {{0, 0, 0}, {2, .3, 1}, {1, -.5, 2}, {-1, .5, 2},
   {-2, .3, 1}, {0, -1.5, 0}, {1, 1, 2}, {0, 0, 4}, {-1, -1, 2}, {0, 0, 0}};
Graphics3D[{Arrow[Tube[BSplineCurve[tref], .15]]}, Boxed → False,
  ViewPoint → {0, -60, 30}, PlotRange → All, ImageSize → 380] // Timing
```

Future Plans

I am interested in producing more pictures of knots with higher number of crossings.

Thank You!