Tuesday Feb 3, hour 13: Quiz 4, Choose and Effective Notation

February-03-15 9:26 AM

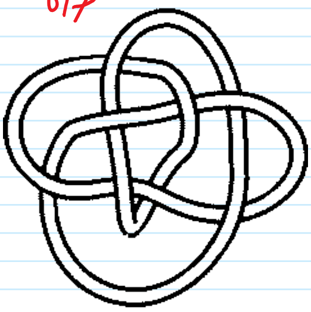
Quiz, then discussion, then

discussion of the handout, starting from the kauffman bracket.

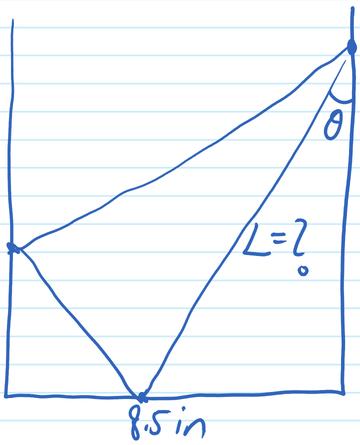
$$\langle \underbrace{\bigcirc \cdots \bigcirc}_{k} \rangle = (-A^2 - A^{-2})^k, \qquad \langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle)(\rangle$$

The Trefoil

8_{17}



Larson's 1.5.7:



Section 1.5:

Additional Examples

1.1.10, 2.5.10, 3.2.15, 3.3.11, 3.3.28, 3.4.2, 3.4.4, 4.1.5, 6.4.2, 7.2.4, 8.1.15, 8.2.3, 8.2.17. Also, see Sections 2.5 (Recurrence Relations), 3.2 (Modular Arithmetic), 3.4 (Positional Notation), 8.3 (Vector Geometry), 8.4 (Complex Numbers in Geometry).

1.1.10. A well-known theorem asserts that a prime $p > 2$ can be written as a sum of two perfect squares ($p = m^2 + n^2$, with m and n integers) if and only if p is one more than a multiple of 4. Make a conjecture concerning which primes $p > 2$ can be written in each of the following forms, using (not necessarily positive) integers x and y: (a) $x^2 + 16y^2$, (b) $4x^2 + 4xy + 5y^2$. (See 1.5.10.)

2.5.10 (Josephus problem). Arrange the numbers $1, 2, \ldots, n$ consecutively (say, clockwise) about the circumference of a circle. Now, remove number 2 and proceed clockwise by removing every other number, among those that remain, until only one number is left. (Thus, for $n = 5$, numbers are removed in the order 2, 4, 1, 5, and 3 remains alone.) Let $f(n)$ denote the final number which remains. Show that

$$f(2n) = 2f(n) - 1,$$
$$f(2n+1) = 2f(n) + 1.$$

(This problem is continued in 3.4.5.)

3.2.15.

(a) Prove that no prime three more than a multiple of four is a sum of two squares. (Hint: Work modulo 4.)
(b) Prove that the sequence (in base-10 notation)

$$11, 111, 1111, 11111, \ldots$$

contains no squares.
(c) Prove that the difference of the squares of any two odd numbers is exactly divisible by 8.
(d) Prove that $2^{70} + 3^{70}$ is divisible by 13.
(e) Prove that the sum of two odd squares cannot be a square.
(f) Determine all integral solutions of $a^2 + b^2 + c^2 = a^2b^2$. (Hint: Analyze modulo 4.)

soln of (b):

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In[6]:= Sort[{10, 1}.Take[IntegerDigits[#^2], -2] & /@ Range[4, 99]]

Out[6]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 4, 4, 4, 9, 9, 9, 16, 16, 16, 16, 21,
  21, 21, 21, 24, 24, 24, 24, 25, 25, 25, 25, 25, 25, 25, 25, 25, 25, 29,
  29, 29, 29, 36, 36, 36, 36, 41, 41, 41, 41, 44, 44, 44, 44, 49, 49, 49,
  49, 56, 56, 56, 56, 61, 61, 61, 61, 64, 64, 64, 64, 69, 69, 69, 69, 76,
  76, 76, 76, 81, 81, 81, 81, 84, 84, 84, 84, 89, 89, 89, 89, 96, 96, 96, 96}
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3.311. Prove that there are an infinite number of primes of the form $6n-1$.

Solution. First, notice that if p is a prime number larger than 3, then either $p \equiv 1 \pmod 6$ or $p \equiv -1 \pmod 6$. [If $p \equiv 2 \pmod 6$, for example, then $p = 6k+2$ for some k, which implies that p is even, a contradiction. A similar argument works for $p \equiv 3 \pmod 6$ or $p \equiv 4 \pmod 6$.]

Now suppose there are only a finite number of primes of the form $6n-1$. Consider the number $N = p! - 1$, where p is the *largest* prime of the form $6n-1$. Write N as a product of primes, say

$$N = p! - 1 = p_1 p_2 \cdots p_m. \tag{1}$$

Observe that each of the primes p_k is larger than p. For, if $p_k \leqslant p$ then equation (1) shows that p_k divides 1, an impossibility. Since p is the largest prime congruent to -1 modulo 6, it follows that $p_k \equiv 1 \pmod 6$ for each k.

If we now consider equation (1) modulo 6, we find that

$$p! - 1 \equiv 1 \pmod 6,$$

or equivalently,

$$p! \equiv 2 \pmod 6.$$

But this is clearly impossible, since $p! \equiv 0 \pmod 6$. Therefore, there must be an infinite number of primes of the form $6n-1$.

3.3.28.

(a) Suppose there are only a finite number of primes of the form $6n-1$; call them $p_1, \ldots, p_k$. Reach a contradiction by considering $N = (p_1 \cdots p_k)^2 - 1$.
(b) Prove that there are an infinite number of primes of the form $4n-1$.

3.4.2. Does $[\![x]\!] + [\![2x]\!] + [\![4x]\!] + [\![8x]\!] + [\![16x]\!] + [\![32x]\!] = 12345$ have a solution?

3.4.4. An (ordered) triple (x_1, x_2, x_3) of positive irrational numbers with $x_1 + x_2 + x_3 = 1$ is called *balanced* if each $x_i < \frac{1}{2}$. If a triple is not balanced, say if $x_j > \frac{1}{2}$, one performs the following "balancing act":

$$B(x_1, x_2, x_3) = (x_1', x_2', x_3'),$$

where $x_i' = 2x_i$ if $i \neq j$ and $x_j' = 2x_j - 1$. If the new triple is not balanced, one performs the balancing act on it. Does continuation of this process always lead to a balanced triple after a finite number of performances of the balancing act?

4.1.5.

(a) If a and b are consecutive integers, show that $a^2 + b^2 + (ab)^2$ is a perfect square.
(b) If $2a$ is the harmonic mean of b and c (i.e., $2a = 2/(1/b + 1/c)$), show that the sum of the squares of the three numbers a, b, and c is the square of a rational number.
(c) If N differs from two successive squares between which it lies by x and y respectively, prove that $N - xy$ is a square.

6.4.2. P is an interior point of the angle whose sides are the rays OA and OB. Locate X on OA and Y on OB so that the line segment XY contains P and so that the product of distances $(PX)(PY)$ is a minimum.

(Solution on absolute page 220).

7.2.4. Suppose that $x_i > 0$, $i = 1, 2, \ldots, n$ and let $x_{n+1} = x_1$. Show that

$$\sum_{i=1}^{n}\left(\frac{x_{i+1}}{x_i}\right) \leqslant \sum_{i=1}^{n}\left(\frac{x_i}{x_{i+1}}\right)^n.$$

Sol at ap 263.

8.1.15. In the following figure, CD is a half chord perpendicular to the diameter AB of the semicircle with center O. A circle with center P is inscribed as shown in Figure 8.13, touching AB at E and arc BD at F.

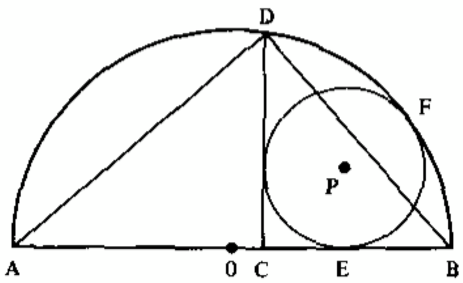

Figure 8.13.

Prove that $\triangle AED$ is isosceles. (Hint: Label the figure and make good use of the Pythagorean theorem.)

8.2.3. A straight line cuts the asymptotes of a hyperbola in points A and B and the curve in points P and Q. Prove that $AP = BQ$.

Sol at ap 306.

8.2.17. Prove that the graph of a cubic equation is symmetric about its point of inflection. (Note: If the conic equation is $f(x) = ax^3 + bx^2 + cx + d$, the x-coordinate of the inflection point is $-b/3a$.)