15-344 Combinatorics on Sep 29 - hours 7-8: Planarity, Euler and Hamilton Circuits

Thursday, September 17, 2015 7:59 PM

Our class' web page is a connected directed graph. Keep it that way!

Read Along: Sects 1.4, 2.1, 2.2.

Thm $K_{3,3}$ & K_5 are not planar.

Def For a plane graph, $\chi(G) := V - e + f$
"The Euler characteristic"

Thm If G is a connected plane graph, then
$\chi(G) = 2$

Correction to proof: "WLOG, G has no deg 1 or 2 verts"
should be "WLOG, G has no degree 1 verts"
Where was connectivity used? on board

Corollary If G is a connected planar graph, $e \leq 3V - 6$

Proof $3f \leq 2e$ so $3f + p = 2e$ $p \geq 0$

$V - e + f = 2 \Rightarrow V - e + \frac{2e - p}{3} = 2$

so $V - \frac{1}{3}e = 2 + \frac{p}{3}$ so $3V - e = 6 + p$ or $3V - e \geq 6$ $\square$

Now if K_5 was planar $3 \cdot 5 - 10 \geq 6$ $\Rightarrow\Leftarrow$.

Kuratowski's Thm

Euler & Hamilton circuits.

The Königsberg Bridges problem

Multigraphs; An Euler cycle.

Thm (Euler, 1736) A multigraph has an Euler cycle iff it is connected and all vertices have an even degree.

Application to Königsberg.

(no edge repeated)

Proof $\Rightarrow$) obvious

$\Leftarrow$ Consider a maximal simple cycle C in G. If it passes through all edges, we're done. otherwise, let v_0 be a vertex along C that is incident to an edge not in C [must exist!]. Find a simple cycle C' in $G-C$, starting and ending in v & consider CC'....

Thm A connected multigraph G has an Euler trail iff all vertices in it are of even degree, except exactly two.

done line

Def Hamilton circuit - a cycle that visits every vertex exactly once. Hamilton path: ...

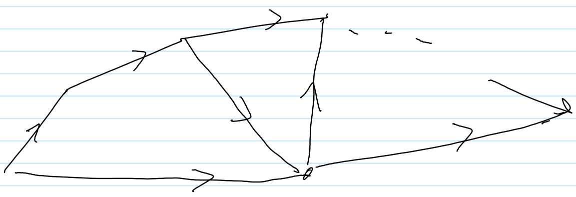
Example: City tour of Europe.

Provably no excellent way to solve?

Def A "tournament" is a directed graph obtained from K_n by choosing a direction for each edge.

Thm Every tournament has a Hamilton path.

proof



Problem You are facing n on/off switches, presently all "off", and you know that in exactly one "code" configuration, the door opens and you can escape. What's the least number of switches necessary to cover all possibilities?

Def A Gray code is a function $g:\{0,\dots,2^n-1\} \longrightarrow \{0,1\}^n$ s.t. $\forall i$

$f(i)$ & $f(i+1)$ differ by at most one digit.

Thm Gray code $\Leftrightarrow$ soln to problem $\Leftrightarrow$ Hamilton path in the n-dim cube graph C_n.

Thm C_n has a Hamilton path.

consider moving my office hours!