15-344 Combinatorics on Oct 29, hour 21: Dijkstra's Algorithm, minimal spanning trees

Tuesday, October 13, 2015 10:38 AM

Follow the handout!

Dijkstra's Algorithm

Problem. Given a connected network $G = (V, E)$; $w\colon E \to \mathbb{R}_{\geq 0}$ and two vertices $a, z \in V$, find the w-shortest path from a to z.

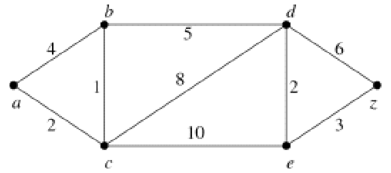
Dijkstra's Algorithm. Throughout, we'll have a decomposition $V = C \sqcup U$ of V into two disjoint sets, C for Confirmed / Confident and U for Unknown, a function $m\colon V \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ for "min so far", and a partially defined function $b\colon V \to V$ for "backtracking". As we run, the set C will grow and the set U will shrink. We stop when $z \in C$, and then $m(z)$ will be "the time to z" and $(z, b(z), b(b(z)), \ldots)$ will be the path from a to z, going backwards.

Initialization. Set $C := \{a\}$, $U := V \setminus \{a\}$, $m(a) := 0$, and $b(a) := ☺$. Also, if x is adjacent to a set $m(x) := w(ax)$ and $b(x) := a$, and for all other x set $m(x) := \infty$ and $b(x) = \text{undef}$.

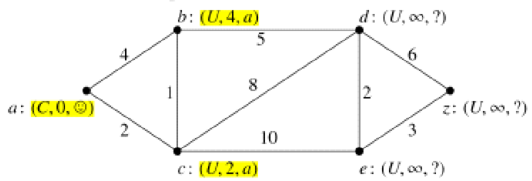
missed view

Iteration. Let x_0 be where m attains its minimum on U (breaking ties arbitrarily), move x_0 from U to C, and for each neighbor $y \in U$ of x_0, if $m(x_0) + w(x_0 y) \geq m(y)$, do nothing. Else set $m(y) := m(x_0) + w(x_0 y)$ and $b(y) := x_0$.

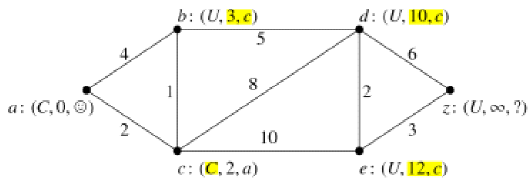
Example. Solve the shortest path problem for the network:



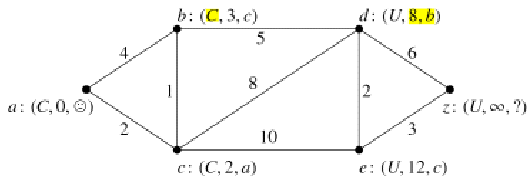
Initialization / Step 0. Notation: $(C/U, m, b)$



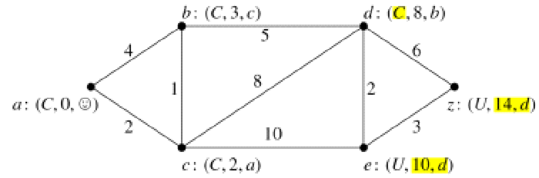
Step 1. $x_0 = c$ and



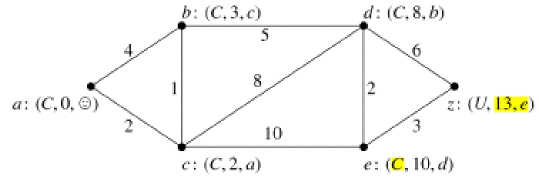
Step 2. $x_0 = b$ and



Step 3. $x_0 = d$ and



Step 4. $x_0 = e$ and



Step 5, end. $x_0 = z$ so we are done; the minimal path length is $m(z) = 13$, and the path, going backwards, is $z \xrightarrow{b} e \xrightarrow{b} d \xrightarrow{b} b \xrightarrow{b} c \xrightarrow{b} a$.

Proof that the algorithm works. The inductive assertion is "after each step, for each $x \in C$ the minimal path length to x is $m(x)$ and the stop before x is $b(x)$; for each $x \in U$ for which $m(x) < \infty$, the minimal path length where all stops but the last are in C is $m(x)$ and the stop before x (in such a path) is $b(x)$".

also, $m(C) < m(U)$

Efficiency estimate. For concreteness, take $|V| = 1{,}000{,}000$ and assume that the maximal degree of a vertex is 7.

Very naively, the search for x_0 is fast and we need about $7{,}000{,}000$ operations in total.

Less naively, the search for x_0 takes about $500{,}000$ operations, so our total is $1{,}000{,}000 \times (7 + 500{,}000)$.

Cleverly, instead of a search, we maintain an ordered table of the values of m. Updating the table takes about $\log_2(500{,}000) \sim 20$ operations, so the total number of operations required is about $1{,}000{,}000 \times (7 + 7 \times 20)$, a feasible number.

Finding a minimal spanning tree.

Kruskal's Algorithm. Start with $T = \emptyset$; repeatedly add to T the cheapest edge that does not form a circuit with edges already in T.

Prim's Algorithm. Start with $T = \emptyset$; repeatedly add to T the cheapest edge that connects T and the complement T^c of T.

Read Along. Section 4.1 and 4.2, and all of section 5.
HW5 will be on the web by midnight tonight, October 29.
No Dror office hours on Tuesday November 3, sorry.

... then prove that Prim is minimal.

I should have pre-written it line by line.

done line, though proof was rushed.

Then basic enumeration, if time, following

(not on web)
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Basic Enumeration

Example 1: Rolling Dice

Two dice are rolled, one green and one red. Each die has faces numbered 1 through 6.

(a) How many different outcomes of this procedure are there?

(b) What is the probability that there are no doubles (not the same value on both dice)?

Example 2: Arranging Books

There are five different Spanish books, six different French books, and eight different Transylvanian books. How many ways are there to pick an (unordered) pair of two books not both in the same language?

Example 3: Sequences of Letters

How many ways are there to form a three-letter sequence using the letters a, b, c, d, e, f **(a)** with repetition of letters allowed? **(b)** without repetition of any letter? **(c)** without repetition and containing the letter e? **(d)** with repetition and containing e?

Example 4: Nonempty Collections

How many nonempty different collections can be formed from five (identical) apples and eight (identical) oranges?

Example 1: Ranking Wizards

How many ways are there to rank n candidates for the job of chief wizard? If the ranking is made at random (each ranking is equally likely), what is the probability that the fifth candidate, Gandalf, is in second place?

Example 2: Arrangements with Repeated Letters

How many ways are there to arrange the seven letters in the word SYSTEMS? In how many of these arrangements do the three Ss appear consecutively?

Example 3: Binary Sequences

How many different 8-digit binary sequences are there with six 1s and two 0s?

Example 4: Poker Probabilities

(a) How many 5-card hands (subsets) can be formed from a standard 52-card deck?

(b) If a 5-card hand is chosen at random, what is the probability of obtaining a flush (all five cards in the hand are in the same suit)?

(c) What is the probability of obtaining three, but not four, Aces?

Example 4: Nonempty Collections

How many nonempty different collections can be formed from five (identical) apples and eight (identical) oranges?

Example 2: (continued) Arrangements with Repetitions

How many arrangements of the seven letters in the word SYSTEMS have the E occurring somewhere before the M? How many arrangements have the E somewhere before the M and the three Ss grouped consecutively?

Example 6: Counting Defective Products

A manufacturing plant produces ovens. At the last stage, an inspector marks the ovens A (acceptable) or U (unacceptable). How many different sequences of 15 As and Us are possible in which the third U appears as the twelfth letter in the sequence?

Example 7: Probability of Repeated Digits

What is the probability that a 4-digit campus telephone number has one or more repeated digits?