15-344 Combinatorics on Nov 26, hour 31: The Catalan Numbers

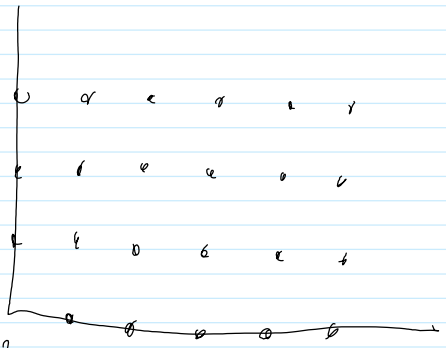
Monday, November 23, 2015 7:11 AM

Evaluations responses: 10/168 https://course-evals.utoronto.ca/blue/a.aspx?l=591_1709_AAAAAAABgaM

Agenda: Catalan, generating functions.

Read Along: Your notes.

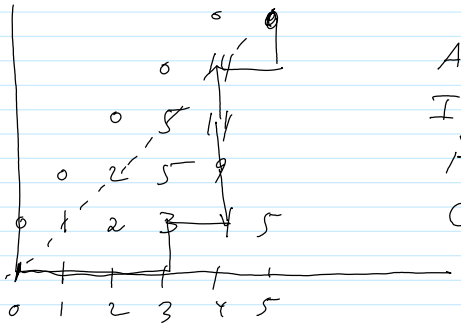
HW9 on web by midnight



Reminder A soccer match ends with the score (n,n). How many scoring histories could the game have had? What if the final score was (n,m)?

on board↑

1. What if it is known that team A was never behind?



Ans: I dunno. call it C_n. As $n=0,1,\dots$, C_n is

0	1	2	3	4	5
1	1	2	5	14	42

2. How many sequences of n a's & n b's are there, in whose initial segments the a's always outnumber the b's?

3. How many ways to compute the product $A_1 \dots A_n$ of n matrices? Ans: C_{n-1}

(an efficiency story)

(the $n!$ story)

(the "pretty little girls school story")

4. How many ways to triangulate a convex n-gon?

done line

Solve using André's reflection method.

The recursion: $C_0 = 1$

On an A-dominated path from $(0,0)$ to (n,n), consider the first tie (k,k), after $(0,0)$:

For $n>0$, $C_n = \sum_{k=1}^{n} C_{k-1} \cdot C_{n-k} = \sum_{k=0}^{n-1} C_k \cdot C_{n-k-1}$

examples, to

$$C_6 = (1\cdot 42 + 1\cdot 14 + 2\cdot 5)\times 2 = 132$$

The generating function $f_C = \sum_{n=0}^{\infty} C_n x^n$:

$$f^2 = \sum_{n\geq 1} C_n x^{n-1} \quad \text{so} \quad xf^2 = f-1$$

so $xf^2 - f + 1 = 0$ so

$$f = \frac{1 \pm \sqrt{1-4x}}{2x} = \frac{1-\sqrt{1-4x}}{2x}$$