15-344 Combinatorics on Dec 1, hours 32-33: More Catalan Numbers

Monday, November 23, 2015 7:11 AM

Evaluations responses: 26/167

Gaurav's a bit sick - no TA office hours today.

https://course-evals.utoronto.ca/blue/a.aspx?l=591_1709_AAAAAAABgaM

Agenda: More Catalan, partitions (if time)

Read Along: Your notes, sec 6.3

					0		
				0	42		
			0	14	42		
		0	5	14	28	48	
	0	2	5	9	14	20	27
0	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1

$n=$ 0 1 2 3 4 5

$C_n=$ 1 1 2 5 14 42

1. In how many soccer histories ending (n,n), team A is never behind?
2. In how many arrangements of $n\times a$ & $n\times b$, every initial string has at least as many a's as b's
3. How many ways to multiply $n+1$ matrices, in order?
4. How many triangulations of an $(n+2)$-gon?

on board.

Sol'n using André's reflection method.

The recursion: $C_0=1$

On an A-dominated path from $(0,0)$ to (n,n), consider the first tie (k,k), after $(0,0)$:

For $n>0$, $C_n=\sum_{k=1}^{n} C_{k-1}\cdot C_{n-k}=\sum_{k=0}^{n-1} C_k\cdot C_{n-k-1}$

examples, to

$C_6=(1\cdot 42+1\cdot 14+2\cdot 5)\times 2=132$

The generating function $f_C=\sum_{n\geq 0} C_n x^n$:

$$f^2=\sum_{n\geq 1} C_n x^{n-1} \quad \text{so} \quad xf^2=f-1$$

so $xf^2-f+1=0$ so

$$f=\frac{1\pm\sqrt{1-4x}}{2x}=\frac{1-\sqrt{1-4x}}{2x}$$

Recall that for $n\in\mathbb{Z}_{\geq 0}$,

$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k=\sum_{k=0}^{\infty}\binom{n}{k}x^k$$

where $\binom{n}{k}=\frac{n(n-1)(n-2)\cdot\ldots\cdot(n-k+1)}{k!}$

__Thm__ For any $\alpha\in\mathbb{R}$, if $|x|<1$ then

$$(1+x)^\alpha=\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k, \text{ where } \binom{\alpha}{k}:=\frac{\alpha\cdot(\alpha-1)\cdot\ldots\cdot(\alpha-k+1)}{k!}$$

& $\binom{\alpha}{0}:=1$.

Half proof (no convergence) using Taylor's thm

In particular,

$$\sqrt{1+y}=(1+y)^{1/2}=\sum_{k\geq 0}\binom{1/2}{k}y^k=\sum_{k\geq 0} b_k$$

Aside $b_k=\binom{1/2}{k}=\frac{1}{k!}\left(\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-k+1\right)\right)$

$$=\frac{1}{k!}\left(\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}\cdot\frac{-7}{2}\cdot\ldots\cdot\frac{\overset{-(2k-3)}{\cancel{1-2k+2}}}{2}\right)$$

$$=\frac{(-1)^{k-1}}{2^k k!}(1\cdot 3\cdot 5\cdot\ldots\cdot(2k-3))$$

done line

$$=\frac{(-1)^{k-1}}{2^k k!}\frac{(2k-2)!}{2\cdot 4\cdot\ldots\cdot(2k-2)}=\frac{(-1)^{k-1}}{2^k k!}\frac{(2k-2)!}{2^{k-1}(k-1)!}$$

$$= \frac{(-1)^{k-1}}{2^{2k-1}k} \cdot \binom{2k-2}{k-1} =: b_k \quad w/\ b_0 = 1$$

So $$\sqrt{1-4x} = 1 + \sum_{k>0} \frac{(-1)^{k-1}}{2^{2k-1}k} \binom{2k-2}{k-1} \cdot (-4)^k x^k$$

$$= 1 + \sum_{k>0} \frac{-2}{k} \binom{2k-2}{k-1} x^k = 1 - 2\sum_{k>0} \frac{1}{k} \binom{2k-2}{k-1} x^k$$

So $$\frac{1-\sqrt{1-4x}}{2x} = \sum_{k>0} \frac{1}{k} \binom{2k-2}{k-1} x^{k-1}$$

$$= \sum_{n\geq 0} \frac{1}{n+1} \binom{2n}{n} x^n \quad !$$

Def A "partition" of n is a way of dividing n identical objects into some number of baskets, whose order is immaterial. P_n is the number of such partitions.

Example: $P_5 = 7$.

5
$4+1$
$3+2$
$3+1+1$
$2+2+1$
$2+1+1+1$
$1+1+1+1+1$

Q What is f_p? I.e., what is $\sum_n P_n x^n$?

Sol'n

$$f_p = (1+x+x^2+\dots)(1+x^2+x^4+x^6+\dots)\cdots$$

$$= \prod_{k=1}^{\infty} \frac{1}{(1-x^k)}$$