UNIVERSITY OF TORONTO

Faculty of Arts and Sciences

DECEMBER EXAMINATIONS 2015

Math 344H1 Combinatorics — Final Exam

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Solve 5 of the following 7 questions. If you solve more than 5 questions indicate very clearly below which are the ones that you want marked, or else a random one will be left out. The questions are of equal value of 20 points each.

Duration. You have 3 hours to write this exam.

Allowed Material. None.

Good Luck!

I should have not put this table here.

Circle the 5 questions that you want marked — otherwise some arbitrary 5 will be marked:

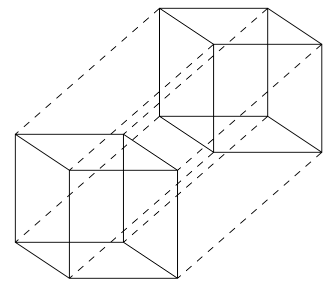
1	/20	5	/20
2	/20	6	/20
3	/20	7	/20
4	/20		
Total	/100		

Solve 5 of the following 7 problems. Each problem is worth 20 points. You have three hours.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Tip. Neatness, cleanliness and organization count, here and everywhere else!

Problem 1. Let G be an arbitrary graph, and for a natural number n let C_n denote the graph of vertices and edges of the n dimensional cube (the case $n = 4$ is depicted on the right).



1. (6 points) Define "an Euler circuit" and "a Hamilton circuit" in G.
2. (7 points) Prove that for $n > 1$, C_n always has a Hamilton circuit.
3. (7 points) For which values of n does C_n have an Euler circuit?

Tip. Quote any theorem you use!

Problem 2. (Should have started "define a spanning tree".) Prove: If G is a connected graph and T and T' are spanning trees in G, then there exists a sequence $T_0, T_1, T_2, \dots, T_n$ of spanning trees in G such that $T_0 = T$, $T_n = T'$, and for every $1 \le k \le n$, the tree T_k is obtained from T_{k-1} by adding one edge and removing one edge.

Evaluation. 3/20. Reasonable writeup of something else. 3/20. Understood what's a "spanning tree". 5/20. Construction that does not make the intermediate steps be spanning trees. −6. No proof that intermediate steps are spanning trees. −6. Moves are not "directed towards goal". −5. Correct, but very loose writeup. −3. Described a process but not why it terminates.

Problem 3. How many ways are there to place 9 rings on the four fingers of your right hand (excluding the thumb) if

1. (6 points) the rings are identical? Ans. $\binom{9+3}{3}$.
2. (6 points) the rings are different and the order of the rings on a finger does not matter? A. 4^9. Evaluation. −4 for 9^4.
3. (8 points) the rings are different and the order of rings on a finger does matter? $9!\binom{9+3}{3} = \frac{12!}{3!}$.

Tip. Answers in the form "$\binom{7}{3} \cdot 9 \cdot 7 \cdot 5$" are acceptable (if correct) — there is no need to find the numerical value of such an expression.

Problem 4. Let C_n be the number of "soccer histories" that start at $(0, 0)$, end in (n, n), and in which the first team is never behind. Show by whatever means you wish that $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Evaluation. 4/20. Built the "Catalan triangle" and nothing more. −12. No justification for bad-path formula.
−8. Referred correctly to "André's reflection" without explaining what it is.
−6. So specification of what part of the path to reflect; not clear why comes to $(n-1, n+1)$.
−4. "Reflect about a diagonal" without specifying which one.
−4. Correct reflection but no statement of what it proves.

Tip. In math-talk, "show" means "prove".

Problem 5.

1. (5 points) Show that if f is the generating function of a sequence a_k, namely $f(x) = \sum_{k=0}^{\infty} a_k x^k$, then xf' is the generating function of the sequence ka_k.

2. (5 points) For some fixed natural number n, find the generating function of the sequence $k\binom{n}{k}$.

   Answer. $nx(1+x)^{n-1}$.

   Evaluation. -2 incorrect/no differentiation.

3. (5 points) Likewise find the generating function of the sequence $k^2\binom{n}{k}$.

   Answer. $nx\left((1+x)^{n-1} + x(n-1)(1+x)^{n-2}\right) = nx(1+nx)(1+x)^{n-2}$

   Evaluation. -2 incorrect/no differentiation.

4. (5 points) Compute the sum $\sum_{k=0}^{n} k^2\binom{n}{k}$. Answer. $n(n+1)2^{n-2}$

Problem 6. Consider Pascal's triangle PT, as partially depicted on the right.

1. (10 points) Show that the sum of every vertical column of numbers in PT (should have: "...which starts at the 1 at the top of a column") is the number below it and to the right. (For example, for the column surrounded by a rectangle on the right, the sum is $1+5+15+35 = 56$, as underlined below it and to the right).

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	$\boxed{1}$				
1	5	10	10	$\boxed{5}$	1			
1	6	15	20	$\boxed{15}$	6	1		
1	7	21	35	$\boxed{35}$	21	7	1	
1	8	28	56	70	$\underline{56}$	28	8	1

Evaluation. 3/10 Checked all possible sub-examples for the displayed PT; 5/20 if for both parts.

-2. Column did not start at top, then correct solution for as if it did.

-5. Derived correct algebraic formula, but did not prove it.

2. (10 points) Show that the sum of every "upwards and to the right" diagonal chain of numbers in PT is a Fibonacci number. (For example, for the diagonal surrounded by an ellipse on the right, the sum is $1+5+6+1 = 13$, which is F_6).

Evaluation. 4/10. Meant the right thing, completely deficient writeup.

Problem 7.

1. (15 points) Let F_n be the Fibonacci sequence, defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ (so $F_n = (1, 1, 2, 3, 5, 8, 13, \ldots)$). Using whatever means you wish, find an explicit formula for the generating function of this sequence, $f(x) := \sum_{n=0}^{\infty} F_n x^n$.

Evaluation. 5/15. Found formula for F_n, but not for $f(x)$. -3 wrong numerator, right denominator.

2. (5 points) Let G_n be the sequence defined by $G_0 = G_1 = 1$, $G_2 = 2$, and $G_n = G_{n-1} + G_{n-2} + G_{n-3}$ for $n \geq 3$ (so $G_n = (1, 1, 2, 4, 7, 13, 24, \ldots)$). Using whatever means you wish, find an explicit formula for the generating function of this sequence, $g(x) := \sum_{n=0}^{\infty} G_n x^n$.

Tip. In math exams, "find" means "find and explain how you found".

Evaluation. 10/20. Wrote $(x+x^2)^k$ and $(x+x^2+x^3)^k$, no summation.

Good Luck!