240AlgebraI-141112 Hours 28-29: Computing the rank, inverting matrices

November-10-14 11:04 AM

Office hour today $1^{30} - 2^{30}$.

$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[d, "P"] \\ V' \arrow[u, "Q"] \arrow[r, "T' = PTQ"] & W' \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[\,]_\beta}"'] & W \arrow[d, "{[\,]_\gamma}"] \\ F^n \arrow[r, "A", "T_A"'] & F^m \end{tikzcd}$$

rank T = rank PTQ $\qquad$ rank $[T]_\beta^\gamma$ = rank T

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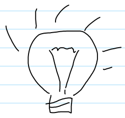
rank A = rank PAQ whenever $P \in M_{m\times m}$ & $Q \in M_{n\times n}$ are invertible.



Look for P & Q that will make PAQ "simpler" than A.

Q1 Which P, Q? $\qquad$ Q2 What's simpler?

Ans 2 $\quad \text{rank}\begin{pmatrix} \underbrace{1 \; 1 \; \ddots \; 1}_{k} & 0 \\ 0 & 0 \end{pmatrix} = \text{rank}\left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right) = k$

Ans 1 Examples of "good" P/Q: $\qquad$ "elementary matrices"

1. Interchanging rows/columns. $\quad E^1_{i,j} \quad (\xrightarrow{r_i \leftrightarrow r_j})$ or $(\xrightarrow{c_i \leftrightarrow c_j})$
2. Multiplying r/c by a scalar. $\quad E^2_{i,c} \quad (\xrightarrow{r_i *= c})$ or $(\xrightarrow{c_i *= c})$
3. Adding a multiple of one r/c to another. $\quad E^3_{i,j,c} \quad (r_i += c\, r_j)$ or $c_j += c \cdot c_i$

$$E^3_{i,j,c} = \begin{array}{c} \\ i \\ \\ \\ \end{array}\begin{pmatrix} 1 & & & \\ & 1 & c & \\ & & 1 & \\ & & & 1 \end{pmatrix} \atop \qquad\qquad j$$

"row/column reduction"

Thm Every matrix A can be r/c-reduced to a block matrix of the form $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$.

Problem. Find the rank the matrix

$$A = \begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}.$$

Solution. Using (invertible!) row/column operations we aim to bring A to look as close as possible to an identity matrix:

Do	Get	Do	Get
1. Bring a 1 to the upper left corner by swapping the first two rows and multiplying the first row (after the swap) by $1/4$.	$\begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}$	2. Add (-8) times the first row to the third row, in order to cancel the 8 in position 3-1.	$\begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}$
3. Likewise add (-6) times the first row to the fourth row, in order to cancel the 6 in position 4-1.	$\begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix}$	4. With similar column operations (you need three of those) cancel all the entries in the first row (except, of course, the first, which is used in the canceling).	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix}$
5. Turn the 2-2 entry to a 1 by multiplying the second row by $1/2$.	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix}$	6. Using two row operations "clean" the second column; that is, cancel all entries in it other than the "pivot" 1 at position 2-2.	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix}$
7. Using three column operations clean the second row except the pivot.	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix}$	8. Clean up the row and the column of the 4 in position 3-3 by first multiplying the third row by $1/4$ and then performing the appropriate row and column transformations. Notice that by pure luck, the 4 at position 4-5 of the matrix gets killed in action.	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Thus the rank of our matrix is 3.

http://drorbn.net/index.php?title=12-240/Classnotes_for_Tuesday_November_8

Claim $\operatorname{rank} A = \operatorname{rank}(A^T)$

BTW, the meaning of A^T in the world of l.t. is quite intricate.

claim $\operatorname{rank} A = \dim(\text{col-space}(A)) = \dim(\text{row-space}(A))$

Suppose you could row reduce A to I. Find A^{-1}.

$$E_4E_3E_2E_1A = I \Rightarrow A^{-1} = E_4E_3E_2E_1$$

* The hard way.
* The easy way: r.r. $(A \mid I)$

Example: compute $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1}$.

done line

How far can you go with row reduction? (And then with cols? BTW, this is an

How far can you go with row reduction? (And then with col?; BTW, this is an amusing app of associativity)

1. The first non-zero entry in each row ("the pivot") is a 1.
2. In the column of a pivot, all else is 0 [Scan from left to right, to prevent interference]
3. Going down the rows, the pivots are further & further to the right.

"reduced row echelon form" r.r.e.f

Example:
$$\begin{array}{l}\text{pivotal col's: 1, 2, 5; non-pivotal col's: 3, 4, 6}\\ \left[\begin{array}{cccccc}1&0&2&9&0&e\\0&1&-3&7&0&\pi\\0&0&0&0&1&2\\0&0&0&0&0&0\end{array}\right]\begin{array}{l}\left.\begin{array}{l}\\ \\ \\ \end{array}\right\}\text{non-zero rows / Pivotal rows}\\ \}\ \text{zero rows / non-pivotal rows.}\end{array}\end{array}$$

.... And now with col. ops., can reach $\left(\begin{array}{c|c}I_r&0\\\hline 0&0\end{array}\right)$

<u>Claim</u> The rank of a r.r.e.f matrix is the number of pivots / non-zero rows in it.

<u>Claim</u> If A is invertible, its r.r.e.f. is I