240Algebra1-141110 Hour 27: Computing the rank

November-09-14 10:26 AM

HW8 on web!

Office Hour this week Wed $1^{30}-2^{30}$ (not at 2^{30})

Next goals: 1. Compute rank T/A. 2. Compute A^{-1} (when possible)

3. Solve systems of linear eqns.

The good and the bad about "matrix algebra":

Good	Bad
1 $A+B=B+A$, $(A+B)+C=A+(B+C)$ (basically, all works for addition)	1. Addition is defined only for matrices of same dims.
2. $A(B\cdot C)=(A\cdot B)C$; $\exists I$ s.t. $A\cdot I=A$, $IA=A$	2. mult. is defined only if "mid" dimension matches, & produces an output of yet other dims.
3 If $A\cdot A^{-1}=I$, then $A^{-1}A=I$	3. A^{-1} may not exist even if $A\neq 0$. } review!
4. $(A+B)C=AC+BC$; $A(B+C)=AB+AC$	4. Generally, $A\cdot B\neq B\cdot A$, even when both make sense.

start line

Proposition Given $V'\xrightarrow{Q}V\xrightarrow{T}W\xrightarrow{P}W'$ with invertible P&Q, rank $T=$ rank PTQ [enough that Q surjective & P injective]

Pf

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ Q\uparrow & & \downarrow P \\ V' & \xrightarrow[PTQ]{T'=} & W' \end{array}$$

$W\supset im(T)=C$, basis $=(w_i=T(v_i))_{i=1}^{r}$

$W'\supset im(T')=C'$ basis $=(w_i'=P(w_i))_{i=1}^{r}$

Need: 1. $w_i'\in im\,T'$; meaning $\exists v_i'\in V'$ s.t. $w_i'=T'v_i'$

2. w_i' span C'

3. w_i' are lin. indep.

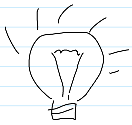
Def If $A\in M_{m\times n}$, let rank $A:=$ rank T_A, where T_A is the "standard" $T_A:F^n\to F^m$

Comment 1 rank $[T]_\beta^\gamma=$ rank T **Pf.**

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ {[\;]_\beta}\downarrow & & \downarrow{[\;]_\gamma} \\ F^n & \xrightarrow[T_A]{A} & F^m \end{array}$$

done line

Comment 2 rank $A=$ rank PAQ whenever $P\in M_{m\times m}$ & $Q\in M_{n\times n}$ are invertible.



Look for P & Q that will make PAQ "simpler" than A.

Q1 Which P, Q? Q2 What's simpler?

Ans 2 $\operatorname{rank}\begin{pmatrix} \underbrace{1 \; \ddots \; 1}_{k} & 0 \\ 0 & 0 \end{pmatrix} = \operatorname{rank}\left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right) = k$

Ans 1 Examples of "good" P/Q: "elementary matrices"

1. Interchanging rows/columns. $E^1_{i,j}$ $\left(\xrightarrow{r_i \leftrightarrow r_j}\right)$ or $\left(\xrightarrow{c_i \leftrightarrow c_j}\right)$
2. Multiplying r/c by a scalar. $E^2_{i,c}$ $\left(\xrightarrow{r_i *= c}\right)$ or $\left(\xrightarrow{c_i *= c}\right)$
3. Adding a multiple of one r/c to another. $E^3_{i,j,c}$ $(r_i += c\, r_j)$ or $c_j += c \cdot c_i$

$$= \begin{array}{c} i \\ \\ \\ j \end{array}\begin{pmatrix} 1 & & & \\ & 1 & c & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

"row/column reduction"

Thm Every matrix A can be r/c-reduced to a block matrix of the form $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$.